

RELATIONS AMONG CHARACTERISTIC CLASSES AND EXISTENCE OF SINGULAR MAPS

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ABSTRACT. We obtain relations among the characteristic classes of a manifold M admitting corank one maps. Our relations yield strong restrictions on the cobordism class of M and also nonexistence results for singular maps of the projective spaces. We obtain our results through blowing up a manifold along the singular set of a smooth map and perturbing the arising non-generic corank one map.

1. INTRODUCTION

Let M be a smooth closed n -dimensional manifold. In [Bott70] it is shown that a subbundle ξ of the tangent bundle TM is tangent to the leaves of a smooth foliation of M (that is, ξ is integrable) only if the ring generated by the real Pontryagin classes of TM/ξ vanishes in dimensions greater than $2(n - \dim \xi)$. The primary purpose of our paper is to prove analogous vanishing theorems about the Stiefel-Whitney and rational Pontryagin classes in the case of “smooth singular fibrations”, i.e. singular maps of M . For $n > k \geq 0$ let Q be a smooth $(n - k)$ -dimensional manifold and let $f: M \rightarrow Q$ be a smooth map. Denote by Σ the set of singular points of f . A point $p \in \Sigma$ is a Σ^{i_1} singularity of f , in notation $p \in \Sigma^{i_1}$, if the rank of the differential df is equal to $n - i_1$ at p . Inductively we define $\Sigma^{i_1, \dots, i_r} \subset M$, where $i_1 \geq \dots \geq i_r \geq 0$, by taking the Σ^{i_r} points of the restriction $f|_{\Sigma^{i_1, \dots, i_{r-1}}}$. A *Morin map* is a smooth map with only $\Sigma^{k+1, 1, \dots, 1, 0}$ singularities (also called A_m -singularities, where $m - 1$ is the number of copies of “1”). In the present paper, we show that the existence of a Morin map from M to Q implies the vanishing of the ideal generated by the differences $w_I(TM) - w_J(TM) \in H^*(TM; \mathbb{Z}_2)$ of monomials of the same degree consisting of Stiefel-Whitney classes of sufficiently high degrees, where I and J run over all the multiindices with length $|I| = |J|$. In particular, we have

Theorem 1.1. *Let k be odd, M^n be orientable, and suppose there exists a Morin map $f: M \rightarrow \mathbb{R}^{n-k}$. Then*

$$\prod_{j=1}^m w_{r_j}(TM) = \prod_{j=1}^m w_{s_j}(TM)$$

for any m and collections r_j, s_j , $j = 1, \dots, m$, which satisfy the conditions $r_j, s_j \geq k + 3$, $j = 1, \dots, m$, and $\sum_{j=1}^m r_j = \sum_{j=1}^m s_j$. If f is a fold map, then the same holds with $r_j, s_j \geq k + 2$, $j = 1, \dots, m$.

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¹After a generic perturbation of f , we can assume that $\Sigma^{i_1, \dots, i_{r-1}}$ is a smooth submanifold of M , see [Boa67].

We prove this in a more general form, see Theorem 3.1. In the proof we proceed by blowing up the source manifold of a Morin map f along the singular set and perturbing $f \circ \pi$, where π is the projection map of the blowup, see Theorems 2.5 and 2.6.

As an application of Theorem 1.1 and Proposition 5.4, we obtain

Theorem 1.2. *Let $n = 2^d + c$ with $0 \leq c < 2^d - 2$, c is odd.*

- (1) *There is no fold map of $\mathbb{R}P^n$ into \mathbb{R}^{n-k} for $1 \leq k+1 < c$.*
- (2) *There is no Morin map of $\mathbb{R}P^n$ into \mathbb{R}^{n-k} for $k+2 < c$ if k is odd.*

For example there is no Morin map of $\mathbb{R}P^{13}$ into \mathbb{R}^{12} , and there is no fold map of $\mathbb{R}P^{11}$ into \mathbb{R}^{10} .

We call a smooth map from M^n to Q^{n-k} a *corank 1 map* if the rank of its differential is not less than $n - k - 1$ at any point of M . About the vanishing of rational Pontryagin classes of TM , we have the analogous result to [Bott70]:

Theorem 1.3. *Suppose M^n admits a corank 1 map into \mathbb{R}^{n-k} . Then the rational Pontryagin class $p_i^{\mathbb{Q}}(TM) \in H^{4i}(M; \mathbb{Q})$ vanishes for $2i > k + 1$.*

For example, there is no corank 1 map of $\mathbb{C}P^n$ into \mathbb{R}^{2n-k} if $\lfloor n/2 \rfloor \geq (k+2)/2$. By Thom transversality and computing the codimension of the Boardman manifolds [Boa67], we have that if $n < 2(k+2)$, then M^n admits corank 1 maps into Q^{n-k} .

Hence for even n , we obtain that $\mathbb{C}P^n$ has a corank 1 map into \mathbb{R}^{2n-k} if and only if $n < k+2$. For odd $n \geq 3$, we do not know whether corank 1 maps exist from $\mathbb{C}P^n$ to \mathbb{R}^{n+2} .

We also obtain results about the cobordism class of the source manifold of a Morin map by combining our relations among characteristic numbers of the source manifold (see Proposition 5.10) with Dold relations.

Theorem 1.4. *Suppose M^n is orientable and admits a fold map into \mathbb{R}^{n-k} . Then*

- (1) *if $k = 1$ and $0 < n \neq 2^a + 2^b - 1$, $a > b \geq 0$, then M is null-cobordant,*
- (2) *if $n > k \geq 5$, k is odd, $k \neq 2^a - 1$, $a \geq 3$, $n - k \neq 1, 3, 7$ and $w_i(TM) = 0$ for $i = 2, \dots, k$, then M is null-cobordant.*

For fold maps into $(n - k)$ -dimensional manifolds with $k = 2^a - 1$, $a > 1$, we have Conjecture 3.20, which we verified for $n \leq 1200$ and $3 \leq k \leq 1023$ by using a computer.

Theorem 1.5. *Suppose M^n is orientable and admits a Morin map into \mathbb{R}^{n-k} . If $n - k = 5, 6$ or $n - k \geq 9$, k is odd and $w_i(TM) = 0$ for $i = 2, \dots, k + 1$, then M is null-cobordant.*

Note that $w_j(TM) = 0$ holds for all $j = 1, \dots, k$ if for example M is k -connected, i.e. all the homotopy groups $\pi_j(M)$ vanish for $1 \leq j \leq k$.

Our results give easy to use criteria for the existence of fold maps, Morin maps and corank 1 maps in general. Up to the present, some practical methods to check the existence of some singular map in general have already been obtained:

- There exists a fold map $f: M \rightarrow Q$ with cokernel $f^*TQ/f^*df(TM)$ being trivial on the singular set if and only if there is a bundle epimorphism $TM \oplus \varepsilon^1 \rightarrow TQ$ [An04, Sae92]. This gives a complete answer to the problem of existence of fold maps with $k \equiv 0 \pmod{2}$ [An04], which can be easily used for further computations when k is even, see for example [SSS10].
- More general versions of this result are deep theorems stating h-principles, which are hard to apply directly and led to criteria using Thom polynomials, see for example [An85, An87, An01].

- There exist fold maps and cusp maps of M into an almost parallelizable manifold only if the Euler characteristic $\chi(M)$ is even, under the assumption that $n - k$ is big enough [SS98]. Refinements of [SS98] include results for Morin maps as well when k is odd [An07, Sad03] but nothing is known when $\chi(M)$ is even.
- For odd k , the self-intersection class of the singular set of a generic corank 1 map f of M into Q is equal to the $(k + 1)/2$ -th Pontryagin class of $TM - f^*TQ$ modulo 2-torsion [OSS03].

The paper is organized as follows. In §2 we present the main results about blowing up the source manifold of a singular map. In §3 we present the main results about the characteristic classes of the source manifold of a singular map. In §4 we prove the statements of §2, and in §5 we prove the statements of §3.

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Conventions. All manifolds henceforth are assumed to be smooth of class C^∞ . Let \mathfrak{N}_n denote the unoriented cobordism group of n -dimensional manifolds. The term “cobordant” refers to unoriented cobordism unless oriented cobordism is specified explicitly. For a finite CW-complex X , $\tilde{K}_\mathbb{R}(X)$ and $K_\mathbb{R}(X)$ denote the reduced and unreduced real K-rings of X , respectively, with $\tilde{K}_\mathbb{R}(X) \subseteq K_\mathbb{R}(X)$. The symbol ε_X^n denotes the trivial n -dimensional bundle over the space X , the lower index “ X ” will be omitted when it is clear from the context. Wherever not stated otherwise, we use the convention that if $\beta < 0$ or $\alpha < \beta$, then the binomial coefficient $\binom{\alpha}{\beta} = 0$.

2. BLOWING UP THE SOURCE MANIFOLD ALONG THE SINGULAR SET

For $n > k \geq 0$, let M be a closed n -manifold and Q be an $(n - k)$ -manifold. It is known that Morin maps are generic corank 1 maps², the singular set of a Morin map of M into Q is an embedded $(n - k - 1)$ -dimensional manifold, the closure of $\Sigma^{k+1,1,\dots,1,0}$ is $\Sigma^{k+1,1,\dots,1}$ with the same number of copies of “1”, and we have 1-codimensional embeddings $\Sigma^{k+1} \supset \Sigma^{k+1,1} \supset \dots$ of closed manifolds. Morin maps with only $\Sigma^{k+1,0}$ and $\Sigma^{k+1,1,0}$ singularities are called cusp maps, while cusp maps with only $\Sigma^{k+1,0}$ singularities are called fold maps. Furthermore, the points of $\Sigma^{k+1,0}$ and $\Sigma^{k+1,1,0}$ are called fold singular points and cusp singular points, respectively. We note that in general a corank 1 map cannot be perturbed to obtain a Morin map.

For an odd $k \geq 1$, let $f: M^n \rightarrow Q^{n-k}$ be a Morin map. We denote the $(k+1)$ -dimensional normal bundle of its singular set $\Sigma = \Sigma^{k+1}$ by ξ . For $0 \leq \lambda \leq (k + 1)/2$ let $\Sigma_\lambda^{k+1,0}$ be the set of index λ fold singular points³ of f . Denote by η the restriction of ξ to $\Sigma_{(k+1)/2}^{k+1,0}$. Then, the normal bundle η has structure group $G(\eta)$ generated by transformations of the form

$$(x_1, \dots, x_{k+1}) \mapsto A(x_1, \dots, x_{k+1}) \text{ with } A \in O\left(\frac{k+1}{2}\right) \times O\left(\frac{k+1}{2}\right)$$

and

$$(x_1, \dots, x_{k+1}) \mapsto (x_{(k+1)/2+1}, \dots, x_{k+1}, x_1, \dots, x_{(k+1)/2}).$$

²In [GG73, Chapter VI §1] it is called 1-generic and corank at most 1 everywhere.

³The index is well-defined if we consider that λ and $k + 1 - \lambda$ represent the same index.

The restriction of f to any fiber of η is left-right equivalent to the saddle singularity

$$(x_1, \dots, x_{k+1}) \mapsto \sum_{i=1}^{(k+1)/2} x_i^2 - \sum_{i=(k+1)/2+1}^{k+1} x_i^2,$$

i.e., to the fold singularity of index $(k+1)/2$.

Note that even if M and Q are oriented, the index $(k+1)/2$ indefinite fold singular set of f can be non-orientable.

Definition 2.1 (Blowup). Let V be an l -dimensional closed submanifold of M^n , and denote the $(n-l)$ -dimensional normal bundle of V by ζ . Let $\text{Bl}_\zeta M$ denote the manifold obtained by blowing up M along V . Let $\text{Bl}_\zeta f$ denote the composition $f \circ \pi$ where $\pi: \text{Bl}_\zeta M \rightarrow M$ is the natural projection.

Remark 2.2. Let $f: M^n \rightarrow Q^{n-k}$ be a generic corank 1 map. Then the singular set Σ is an embedded $(n-k-1)$ -dimensional submanifold of M [Boa67], denote its normal bundle by ζ . We have that the map $\text{Bl}_\zeta f$ is a non-generic corank 1 map and its singular set is $\pi^{-1}(\Sigma)$.

We will use the notations of the following blowup diagram.

$$\begin{array}{ccc} \pi^{-1}(\Sigma) & \xrightarrow{\tilde{i}} & \text{Bl}_\zeta M \\ \downarrow p & & \downarrow \pi \\ \Sigma & \xrightarrow{i} & M \end{array}$$

Note that $\pi^*: H^m(M; \mathbb{Z}_2) \rightarrow H^m(\text{Bl}_\zeta M; \mathbb{Z}_2)$ is injective for all m . Indeed, consider the Gysin map $\pi_!: H^m(\text{Bl}_\zeta M; \mathbb{Z}_2) \rightarrow H^m(M; \mathbb{Z}_2)$. Denote by PD the Poincaré duality map $\text{PD}: H_n(M^n; \mathbb{Z}_2) \rightarrow H^0(M^n; \mathbb{Z}_2)$, then for any $x \in H^m(M; \mathbb{Z}_2)$ we have

$$\begin{aligned} \pi_!(\pi^*(x)) &= \pi_!(\pi^*(x) \cup 1) = x \cup \pi_!(1) = x \cup \text{PD}(\pi_*([\text{Bl}_\zeta M] \cap 1)) = \\ &= x \cup \text{PD}(\pi_*([\text{Bl}_\zeta M])) = x \cup \text{PD}([M]) = x \cup 1 = x. \end{aligned}$$

Definition 2.3 (Morse-Bott map). For $n > k \geq 0$, we call a smooth map $f: P^n \rightarrow Q^{n-k}$ a *Morse-Bott map* if

- (1) the set S_f of singular points of f is the disjoint union $\sqcup_i S_i$ of smooth closed connected submanifolds of P ,
- (2) each component S_i is the total space of a smooth bundle with a connected manifold C_i as fiber,
- (3) for each component S_i there exist λ and l such that $0 \leq \lambda \leq l \leq k+1$ and for each singular point $p \in S_i$ there exist neighborhoods U_1 of p , U_2 of $f(p)$ and diffeomorphisms $u_1: U_1 \rightarrow \mathbb{R}^l \times \mathbb{R}^{k+1-l} \times \mathbb{R}^{n-k-1}$ and $u_2: U_2 \rightarrow \mathbb{R} \times \mathbb{R}^{n-k-1}$ with the following properties:
 - (a) $u_1(p) = 0$, $u_2(f(p)) = 0$,
 - (b) $u_1(U_1 \cap S_f) = \{(x, y, z) \in \mathbb{R}^l \times \mathbb{R}^{k+1-l} \times \mathbb{R}^{n-k-1} : x = 0\}$,
 - (c) for the fiber C_i containing p , $u_1(U_1 \cap C_i) = \{(x, y, z) \in \mathbb{R}^l \times \mathbb{R}^{k+1-l} \times \mathbb{R}^{n-k-1} : x = 0, z = 0\}$, and
 - (d) $u_2 \circ f \circ u_1^{-1}(x, y, z) = (\sum_{i=1}^\lambda -x_i^2 + \sum_{i=\lambda+1}^l x_i^2, z)$.

The index of f at a singular point p is the pair $(\lambda, k+1-l)$ if $\lambda \leq l - \lambda$.

Remark 2.4. Compare Definition 2.3 with [BH04, Morse-Bott Lemma].

Note that for a Morse-Bott map $f: P \rightarrow Q$ the index is well-defined. Let $\Sigma_{(\lambda, k+1-l)}$ denote the set of singular points of f which have index $(\lambda, k+1-l)$, then $\Sigma_{(\lambda, k+1-l)}$ is an $(n-l)$ -dimensional closed submanifold of P . Also note that a Morse-Bott map is a corank 1 map, although it is not necessarily generic or Morin.

For each index $(\lambda, k+1-l)$, let $\tilde{\Sigma}_{(\lambda, k+1-l)}$ denote the set $\Sigma_{(\lambda, k+1-l)}/\sim$ where $p \sim q$ if and only if p and q lie in the same connected fiber C_i for some i . Clearly $\tilde{\Sigma}_{(\lambda, k+1-l)}$ is an $(n-k-1)$ -dimensional manifold and the continuous map $\tilde{f}_{(\lambda, k+1-l)}: \tilde{\Sigma}_{(\lambda, k+1-l)} \rightarrow Q$ determined by the property $f = \tilde{f}_{(\lambda, k+1-l)} \circ q_\sim$, where $q_\sim: \Sigma_{(\lambda, k+1-l)} \rightarrow \tilde{\Sigma}_{(\lambda, k+1-l)}$ is the quotient map, is an immersion.

The cokernel bundle $(f^*TQ/f^*df(TP))|_{S_f}$ can be identified with the pull-back

$$q_\sim^*(\cup_{(\lambda, k+1-l)} \tilde{f}_{(\lambda, k+1-l)}^*)^* \nu,$$

where ν is the normal bundle of the immersion $\cup_{(\lambda, k+1-l)} \tilde{f}_{(\lambda, k+1-l)}: \cup_{(\lambda, k+1-l)} \tilde{\Sigma}_{(\lambda, k+1-l)} \rightarrow Q$, where $(\lambda, k+1-l)$ runs over all the indices of f .

If $\lambda \neq l - \lambda$, then the normal bundle of the immersion $\tilde{f}_{(\lambda, k+1-l)}$ is trivial.

Theorem 2.5. *Let $k \geq 1$ be odd. For a fold map $f: M^n \rightarrow Q^{n-k}$, we can perturb the map*

$$\text{Bl}_\eta f: \text{Bl}_\eta M \rightarrow Q$$

*in a neighborhood of $\pi^{-1}(\Sigma_{(k+1)/2}^{k+1,0})$ so that the perturbed map $\Theta: \text{Bl}_\eta M \rightarrow Q$ is Morse-Bott and the normal bundle of the immersion $\tilde{\Theta}_{(\lambda, k+1-l)}$ is trivial for each index $(\lambda, k+1-l)$. The stable tangent bundle of $\text{Bl}_\eta M$ splits as $T\text{Bl}_\eta M \oplus \varepsilon^1 \cong \zeta^{k+1} \oplus \Theta^*TQ$ for some $(k+1)$ -dimensional vector bundle ζ^{k+1} over $\text{Bl}_\eta M$.*

By extending Theorem 2.5 to Morin maps, we obtain

Theorem 2.6. *Let $k \geq 1$ be odd. Assume there exists a Morin map $f: M^n \rightarrow Q^{n-k}$. Then the stable tangent bundle of $\text{Bl}_\xi M$ splits as*

$$T\text{Bl}_\xi M \oplus \varepsilon^2 \cong \zeta^{k+2} \oplus (\widetilde{\text{Bl}_\xi f})^*TQ$$

for some $(k+2)$ -dimensional vector bundle ζ^{k+2} over $\text{Bl}_\xi M$ and perturbation $\widetilde{\text{Bl}_\xi f}$ of $\text{Bl}_\xi f$.

Remark 2.7. In Theorems 2.5 and 2.6 if Q is stably parallelizable, then the bundles $T\text{Bl}_\eta M$ and $T\text{Bl}_\xi M$ are stably equivalent to $(k+1)$ - and $(k+2)$ -dimensional bundles, respectively.

Remark 2.8. If we blow up M along all the singular set Σ of a fold map, then we can perturb $\text{Bl}_\xi f: \text{Bl}_\xi M \rightarrow Q$ in a neighborhood of $\pi^{-1}(\Sigma_{(k+1)/2}^{k+1,0})$ so that the stable tangent bundle of $\text{Bl}_\xi M$ splits as

$$T\text{Bl}_\xi M \oplus \varepsilon^1 \cong \zeta^{k+1} \oplus (\widetilde{\text{Bl}_\xi f})^*TQ$$

for some $(k+1)$ -dimensional vector bundle ζ^{k+1} over $\text{Bl}_\xi M$ and perturbation $\widetilde{\text{Bl}_\xi f}$ of $\text{Bl}_\xi f$.

3. CHARACTERISTIC CLASSES OF THE SOURCE MANIFOLD

By using the results of §2, we obtain the following relations between the Stiefel-Whitney classes of the source manifold of a Morin map.

Theorem 3.1. *Let k be odd, M be an orientable n -manifold and Q be an orientable $(n-k)$ -manifold. Assume $K \geq 0$ is such that $w_i(TQ) = 0$ for $i > K$, furthermore for any m and*

$j = 1, \dots, m$ let $r_j, s_j \geq k + 3 + K$ be natural numbers such that $\sum_{j=1}^m r_j = \sum_{j=1}^m s_j$. If M admits a Morin map into Q , then

$$(3.1) \quad w_{r_1}(TM) \cdots w_{r_m}(TM) = w_{s_1}(TM) \cdots w_{s_m}(TM).$$

The same holds under the relaxed condition $r_j, s_j \geq k + 2 + K$ if there is a fold map of M into Q .

For example, $w_5(T\mathbb{R}P^{13}) = 0$ and $w_6(T\mathbb{R}P^{13}) \neq 0$, thus there is no Morin map of $\mathbb{R}P^{13}$ into \mathbb{R}^{12} .

Remark 3.2.

- (1) Theorem 3.1 holds also if M and Q are possibly non-orientable and the Morin map of M into Q is a cusp map.
- (2) Note that if $k \geq 0$ and k is even, then (3.1) obviously holds for a fold map if $r_j, s_j \geq k + 2 + K$ since $w_j(TM) = 0$ for $j \geq k + 2 + K$, see [An04].

By Proposition 5.4 and applying the above to maps of the projective spaces $\mathbb{R}P^n$, we obtain

Corollary 3.3. *Let $n = 2^d + m$ with $0 \leq m < 2^d - 2$, where m is odd. There is no Morin map of $\mathbb{R}P^n$ into \mathbb{R}^{n-k} for $k + 2 < m$ if k is odd. There is no fold map of $\mathbb{R}P^n$ into \mathbb{R}^{n-k} for $k + 1 < m$.*

For example, there is no fold map from $\mathbb{R}P^{13}$ to \mathbb{R}^j with $10 \leq j \leq 13$.

Remark 3.4. By [MS74, Corollary 11.15], we obtain the analogous result for closed n -manifolds M with $H^*(M; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^n; \mathbb{Z}_2)$, and also a more general result for any closed manifold whose cohomology ring with \mathbb{Z}_2 coefficients is generated by one element.

Now let us consider Pontryagin classes.

Theorem 3.5. *Let $f: M^n \rightarrow Q^{n-k}$, $n > k \geq 0$, be a smooth map with $\text{rank } df \geq n - k - 1$ and let Q be stably parallelizable. Then the rational Pontryagin classes $p_i^{\mathbb{Q}}(TM) \in H^{4i}(M; \mathbb{Q})$ vanish for $2i > k + 1$.*

Remark 3.6. For $n \geq 2$ the class $p_{[n/2]}(TCP^n)$ is equal to $\binom{n+1}{[n/2]}y$, where y is the standard generator of $H^{4[n/2]}(\mathbb{C}P^n)$ and hence $p_{[n/2]}^{\mathbb{Q}}(TCP^n)$ does not vanish. Hence there is no corank 1 map of $\mathbb{C}P^n$ to a stably parallelizable target Q^{2n-k} if $[n/2] \geq (k+2)/2$. For example, there exists no Morin map from $\mathbb{C}P^2$ to Q^4 , and from $\mathbb{C}P^4$ to Q^7 (cf. [OSS03, Example 4.9]) or to Q^8 (cf. [SS98, Theorems 1.2 and 1.3]), and there exists no Morin map of $\mathbb{C}P^{49}$ to Q^{93} (cf. [OSS03, Remark 4.5]).

Finally, from the viewpoint of K-theory and γ operations, we have the following⁴. Recall that for a finite CW-complex X the geometric dimension $g.\dim(x)$ of an element $x \in \tilde{K}_{\mathbb{R}}(X)$ is the least integer k such that $x + k$ is a class of a genuine vector bundle over X (see e.g. [At61]).

We call a corank 1 map $f: M \rightarrow Q$ *tame* if the 1-dimensional cokernel bundle $\text{coker } df|_{\Sigma}$ of the restriction $df|_{\Sigma}: TM|_{\Sigma} \rightarrow f^*TQ$ is trivial. For example, every fold map is tame for $k \equiv 0 \pmod{2}$ [An04] and it is easy to construct not tame fold maps for odd $k \leq n - 3$, even between orientable manifolds. Also note that a Morse-Bott map f is tame if and only if all the normal bundles of the immersions $\tilde{f}_{(\lambda, k+1-l)}$ are trivial.

⁴We presented these results at the conference “Singularity theory of smooth maps and related geometry”, RIMS, Tokyo, and on the Topology Seminar at Kyushu University, Fukuoka, in 2009 December.

Let M^n and Q^{n-k} be a closed n -manifold and an $(n-k)$ -manifold, respectively.

Proposition 3.7. *The following are equivalent:*

- (1) M admits a tame corank 1 map into Q ,
- (2) there is a fiberwise epimorphism $TM \oplus \varepsilon^1 \rightarrow TQ$.

If Q is stably parallelizable, then (1) and (2) hold if and only if $\text{g.dim}([TM] - [\varepsilon^n]) \leq k + 1$.

For a finite CW-complex X , let $\lambda_t = \sum_{i=0}^{\infty} \lambda^i t^i$, where λ^i are the exterior power operators (for details, see [At61]). Define $\gamma_t = \sum_{i=0}^{\infty} \gamma^i t^i$ to be the homomorphism $\lambda_{t/1-t}$ of $K_{\mathbb{R}}(X)$ into the multiplicative group of formal power series in t with coefficients in $K_{\mathbb{R}}(X)$ and constant term 1. By the above proposition and [At61, Proposition 2.3], we immediately have

Corollary 3.8. ⁵ *If M^n admits a tame corank 1 map into a stably parallelizable Q^{n-k} , then*

- (1) $w_i(TM) = 0$ for $i \geq k + 2$,
- (2) $p_i(TM) = 0$ for $2i > k + 1$,
- (3) $\gamma^i([TM] - [\varepsilon^n]) = 0$ for $i \geq k + 2$.

Remark 3.9. Note that the conditions (1) and (2) may not give strong results in general: for example, all the positive degree Stiefel-Whitney and Pontryagin classes of $\mathbb{R}P^{2^n-1}$ vanish⁶, and if $k + 1 \geq n/2$, then condition (2) is satisfied trivially for any M . In particular cases, though, condition (1) can still give strong results, e.g. all Stiefel-Whitney classes of $\mathbb{R}P^{2^n-2}$ of degree up to $2^n - 2$ are nonzero.

For an integer s let $2^{R(s)}$ be the maximal power of 2 which divides s , and define $\kappa(n) = \max\{0 < s < 2^{n-1} : s - R(s) < 2^{n-1} - n\}$. By using Corollary 3.8 (3) and following a similar argument to [At61], we obtain the following:

Proposition 3.10. *For $n \geq 4$, $\mathbb{R}P^{2^n-1}$ does not admit tame corank 1 maps into \mathbb{R}^{2^n-1-k} for $k \leq \kappa(n) - 2$.*

Remark 3.11. Obviously $s_0 = 2^{n-1} - 2^{\min\{r:r+2^r>n\}}$ satisfies $s_0 + n - R(s_0) < 2^{n-1}$, hence $s_0 \leq \kappa(n)$ and we obtain that $\mathbb{R}P^{2^n-1}$ admits no tame corank 1 map into $\mathbb{R}^{2^{n-1}+2^{\min\{r:r+2^r>n\}}+j}$ for $n \geq 4$ and $j \geq 1$. Also, since $\min\{r : r + 2^r > n\} \leq \lceil \log_2 n \rceil$, the same conclusion holds in the case of the target $\mathbb{R}^{2^{n-1}+2^{\lceil \log_2 n \rceil}+j}$ for $n \geq 4$ and $j \geq 1$. For example, there exists neither a fold map from $\mathbb{R}P^{31}$ to \mathbb{R}^{21+2j} for $0 \leq j \leq 5$ nor a tame corank 1 map from $\mathbb{R}P^{31}$ to \mathbb{R}^{22+2j} for $0 \leq j \leq 4$.

3.1. Cobordism class of the source manifold. For $n \equiv 0 \pmod{4}$, let X^n be a closed oriented n -manifold such that it is null-cobordant as an unoriented manifold and its only nonzero Pontryagin characteristic number is $p_1^{n/4}[X^n] > 0$, which is equal to the minimal even value attainable by manifolds with these properties. We define the following linear subspaces of \mathfrak{N}_n :

- for $n = 2^a$ with $a \geq 2$, \mathfrak{A}^1 is the 1-dimensional space defined by the vanishing of $w_2^{n/2} + w_n$ as well as all monomial Stiefel-Whitney numbers except $w_2^{n/2}$ and w_n . For example, the cobordism class of $(\mathbb{C}P^2)^{n/4}$ generates \mathfrak{A}^1 .
- for $n = 2^{b+1} + 2^b - 1$ with $b \geq 1$, \mathfrak{B}^1 is the 1-dimensional space defined by the vanishing of

⁵Compare with [At61, Proposition 3.2].

⁶We have $w(T\mathbb{R}P^{2^n-1}) = (1+x)^{2^n} = 1 \in \mathbb{Z}_2[x]/x^{2^n} = H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$, where x denotes the generator of $H^1(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$. The natural homomorphism $H^s(\mathbb{R}P^{2^n-1}; \mathbb{Z}) \rightarrow H^s(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$ is an isomorphism for all positive even s . Our claim follows by applying the fact that $p_i \equiv w_{2i}^2 \pmod{2}$.

- all monomial Stiefel-Whitney numbers not of the form $w_{m_1} \cdots w_{m_{2b}}$,
- all monomial Stiefel-Whitney numbers containing w_1 ,
- all pairwise sums of the rest of monomial Stiefel-Whitney numbers.
- for $n = 2^a + 2^b - 1$ with $a \geq b + 2$ and $b \geq 1$, \mathfrak{C}^2 is the two-dimensional space defined by the vanishing of
 - all monomial Stiefel-Whitney numbers which are not either of the form

$$w_{m_1} \cdots w_{m_{2b}} \quad \text{or} \quad w_{m_1} \cdots w_{m_{2a-1}},$$
 - all monomial Stiefel-Whitney numbers containing w_1 ,
 - all pairwise sums of Stiefel-Whitney numbers of the form $w_{m_1} \cdots w_{m_{2a-1}}$ with all $m_j \geq 2$, and all pairwise sums of Stiefel-Whitney numbers of the form $w_{m_1} \cdots w_{m_{2b}}$ with all $m_j \geq 2$.

Theorem 3.12. *Let $n \geq 2$. Assume M is an oriented n -manifold admitting a fold map into a stably parallelizable $(n - 1)$ -manifold. Then either M is oriented null-cobordant or one of the following cases occurs:*

- (1) $n \equiv 0 \pmod{4}$, n is not a power of 2 and M is oriented cobordant to mX^n for some $m \in \mathbb{Z}$.
- (2) $n = 2^a$ for some $a \geq 2$ and either
 - (a) $[M]$ is the nonzero element of \mathfrak{A}^1 , or
 - (b) M is oriented cobordant to mX^n for some $m \in \mathbb{Z}$.
- (3) $n = 2^{b+1} + 2^b - 1$ for some positive integer b and $[M] \in \mathfrak{B}^1$.
- (4) $n = 2^a + 2^b - 1$ for some positive integers a and b , $a \geq b + 2$, and $[M] \in \mathfrak{C}^2$.

Note that in the cases (1) and (2b) M is unoriented null-cobordant. Also, the case (2a) implies $w_n[M] \neq 0$ and can be excluded if $n \neq 2, 4, 8$, see [SS98] and use the fact that a stably parallelizable manifold is almost parallelizable.

Corollary 3.13. *If n is not of the form $2^a + 2^b - 1$ for some integers $a > b \geq 0$ and the orientable n -manifold M has an odd Pontryagin number or a nonzero Stiefel-Whitney number, then M has no fold map into any stably parallelizable $(n - 1)$ -manifold.*

Remark 3.14. If M is a spin manifold, then Corollary 3.13 holds with the relaxed condition $n \neq 2, 4, 8$, see Corollary 5.11.

Remark 3.15. By Theorem 3.12 if an orientable $(4m + 1)$ -manifold M admits a fold map into a stably parallelizable $4m$ -manifold, then the de Rham invariant $w_2 w_{4m-1}[M]$ vanishes. This may suggest further relations of fold maps to surgery theory, see [An01].

Theorem 3.16. *Let $n \geq 2$. There exists a 1-dimensional linear subspace $\mathfrak{D}^1 \leq \mathfrak{N}_n$ such that if M is a possibly non-orientable n -manifold admitting a tame corank 1 map into a stably parallelizable $(n - 1)$ -manifold, then $[M] \in \mathfrak{D}^1$ and $w_1^n[M] = 1$ if M is not null-cobordant.*

Proposition 3.17. *Let $n > k \geq 5$ where k is odd and not of the form $2^a - 1$ for some $a \geq 3$. There exists a 1-dimensional linear subspace $\mathfrak{E}^1 \leq \mathfrak{N}_n$ such that if M is an n -manifold with $w_1(TM) = \cdots = w_k(TM) = 0$ admitting a fold map into a stably parallelizable $(n - k)$ -manifold, then $[M] \in \mathfrak{E}^1$. Additionally, if $[M] \in \mathfrak{E}^1$ and M is not null-cobordant, then $w_n[M] = 1$ is the only nonzero monomial characteristic number of M .*

Again, [SS98] implies that M is null-cobordant if $n - k \neq 1, 3, 7$.

Proposition 3.18. *Let $n > k \geq 1$ and k is odd. There exists a 1-dimensional linear subspace $\mathfrak{F}^1 \leq \mathfrak{N}_n$ such that if M is an n -manifold which admits a Morin map into a stably parallelizable*

$(n - k)$ -manifold and $w_i(TM) = 0$ for $i = 1, \dots, k + 1$, then $[M] \in \mathfrak{F}^1$. Additionally, if $[M] \in \mathfrak{F}^1$ and M is not null-cobordant, then $w_n[M] = 1$ is the only nonzero monomial characteristic number of M .

As before, the case of $w_n[M] \neq 0$ is excluded if $n - k \neq 5, 6$ or $n - k \geq 9$, see [Sad03, SS98].

Remark 3.19. By Theorem 3.1 we can make analogous statements to the above in the case of not stably parallelizable Q as well.

Numerical calculations similar to those of the proof of Theorem 3.12 suggest the following conjecture:

Conjecture 3.20. Let $n > k \geq 2$ and $k = 2^a - 1$, where $a \geq 2$. There exists a 1-dimensional linear subspace $\mathfrak{G}^1 \leq \mathfrak{N}_n$ such that if an n -manifold M with $w_i(TM) = 0$ for $i = 1, \dots, k$ admits a fold map into a stably parallelizable $(n - k)$ -manifold, then we have one of the following cases:

- (1) $n = 2^s$ or $n = 2^s + 1$ with $s \geq a + 1$, and $[M] \in \mathfrak{G}^1$.
- (2) M is null-cobordant.

We verified this conjecture for $n \leq 1200$, $3 \leq k \leq 1023$ with the help of a computer.

4. PERTURBING THE BLOWUP OF A SINGULAR MAP

Proof of Theorem 2.5. Let ν denote the 1-dimensional normal bundle of the immersion

$$f|_{\Sigma_{(k+1)/2}^{k+1,0}} : \Sigma_{(k+1)/2}^{k+1,0} \rightarrow Q.$$

We identify the normal bundle η with a tubular neighborhood of $\Sigma_{(k+1)/2}^{k+1,0}$ in M so that f restricted to η is a composition of

- (1) a (nonlinear) bundle map $\iota : \eta \rightarrow \nu$ having the form

$$(x_1, \dots, x_{k+1}) \mapsto \sum_{i=1}^{(k+1)/2} x_i^2 - \sum_{i=(k+1)/2+1}^{k+1} x_i^2$$

on the unit disk of each fiber of η in suitable local coordinates with

- (2) an immersion $\varphi : \nu \rightarrow Q$.

Under this identification, the points of $\text{Bl}_\eta M$ in a neighborhood of $\pi^{-1}(\Sigma_{(k+1)/2}^{k+1,0})$ are identified with the sets of pairs $[(v, \tilde{v})] := \{(v, \tilde{v}), (v, -\tilde{v})\}$ where v and \tilde{v} are parallel vectors in the same fiber of η and \tilde{v} has length 1. Note that $\text{Bl}_\eta f([(v, \tilde{v})]) = \varphi \circ \iota(v)$ under these identifications.

We define the perturbed map $\Theta : \text{Bl}_\eta M \rightarrow Q$ to agree with $\text{Bl}_\eta f$ outside the π -preimage of the unit disk bundle of η and define Θ by the formula

$$(4.1) \quad \Theta([(v, \tilde{v})]) = \varphi(\iota(v) + \varepsilon(p)\omega(\|v\|)\iota(\tilde{v}))$$

within the π -preimage of this disk bundle of η . Here $\omega : \mathbb{R} \rightarrow [0, 1]$ is a bump function which is equal to 1 around 0 and 0 around 1, $p \in \Sigma_{(k+1)/2}^{k+1,0}$, v and \tilde{v} are in the fiber η_p of η over p , and $\varepsilon(p) = \varepsilon > 0$ is a small real number we will choose later. Note that Θ is well-defined since $\iota(\tilde{v}) = \iota(-\tilde{v})$.

Clearly the differential $d\Theta$ has rank at least $n - k - 1$ outside $\pi^{-1}(\eta)$. From (4.1) it is easy to see that $d\Theta$ has rank at least $n - k - 1$ on $\pi^{-1}(\eta)$ as well since for any small curve $\alpha : p \mapsto [(v, \tilde{v})]_p \in \pi^{-1}(\eta_p)$ going in the fibers $\pi^{-1}(\eta_p)$ of $\pi^{-1}(\eta)$, $p \in \Sigma_{(k+1)/2}^{k+1,0}$, where v, \tilde{v} are fixed, we have that $\Theta(\alpha(p))$ is an immersion. Hence Θ is a corank 1 map. To get the singular

set of $\Theta|_{\pi^{-1}(\eta)}$, we first take a curve $\gamma(t) = [(t\tilde{v}, \tilde{v})]$ in the blowup $\pi^{-1}(\eta_p)$ of a single fiber η_p in $\text{Bl}_\eta M$, $p \in \Sigma_{(k+1)/2}^{k+1,0}$. The composite map

$$\Theta \circ \gamma(t) = \varphi(\iota(t\tilde{v}) + \varepsilon\omega(t)\iota(\tilde{v})) = \varphi((t^2 + \varepsilon\omega(t))\iota(\tilde{v}))$$

has a single critical point at $t = 0$ if $\iota(\tilde{v}) \neq 0$ and ε is small enough. Taking the curve $\delta(s) = [(t_0\tilde{v}_s, \tilde{v}_s)]$ with a fixed t_0 , $\iota(\tilde{v}_0) = 0$ so that it intersects $\{[(v, \tilde{v})] : \iota(\tilde{v}) = 0\}$ transversally the composite map

$$\Theta \circ \delta(s) = \varphi(\iota(t_0\tilde{v}_s) + \varepsilon\omega(t_0)\iota(\tilde{v}_s)) = \varphi((t_0^2 + \varepsilon\omega(t_0))\iota(\tilde{v}_s))$$

has nonzero derivative at $s = 0$. Hence the singular points of $\Theta|_{\pi^{-1}(\eta_p)}$ are contained in $\pi^{-1}(p)$ and the singular points of $\Theta|_{\pi^{-1}(\eta)}$ are contained in $\pi^{-1}(\Sigma_{(k+1)/2}^{k+1,0})$.

Clearly a critical point of $\Theta|_{\pi^{-1}(\eta_p)}$ is a critical point of $\Theta|_{\pi^{-1}(p)}$. In the following, we show that at the critical points of $\Theta|_{\pi^{-1}(p)}$ the composite map $\Theta \circ \gamma(t)$, where $\gamma(t) = [(t\tilde{v}, \tilde{v})]$, has a critical point for $t = 0$. Hence any critical point of $\Theta|_{\pi^{-1}(p)}$ is a critical point of $\Theta|_{\pi^{-1}(\eta_p)}$. The choice of coordinates x_1, \dots, x_{k+1} on η_p identifies $\pi^{-1}(p)$ with the projective space $\mathbb{R}P^k$, and the restriction $\Theta|_{\pi^{-1}(p)}$ is equal to

$$\Theta|_{\pi^{-1}(p)}: [x_1 : \dots : x_{k+1}] \mapsto \varphi\left(\varepsilon\iota\left(\frac{(x_1, \dots, x_{k+1})}{\|(x_1, \dots, x_{k+1})\|}\right)\right).$$

This map is Morse-Bott and has critical points along two copies of $\mathbb{R}P^{(k+1)/2-1}$, which are $\{[x_1 : \dots : x_{(k+1)/2} : 0 : \dots : 0] \in \mathbb{R}P^k\}$ and $\{[0 : \dots : 0 : x_{(k+1)/2+1} : \dots : x_{k+1}] \in \mathbb{R}P^k\}$, and it is easy to see that both critical loci have index $(0, (k+1)/2-1)$. Therefore $\Theta|_{\pi^{-1}(\eta_p)}$ is a Morse-Bott map with indices $(1, (k+1)/2-1)$ and with this two copies of $\mathbb{R}P^{(k+1)/2-1}$ as singular set. Hence Θ is a Morse-Bott map with the corresponding indices. We also have that the singular set of $\Theta|_{\pi^{-1}(\eta)}$ is a fiber bundle with the singular sets of $\Theta|_{\pi^{-1}(\eta_p)}$, $p \in \Sigma_{(k+1)/2}^{k+1,0}$, as fibers. It is easy to see that the $(n-k-1)$ -manifold $q_\sim(\Sigma_{(1,(k+1)/2-1}) \cap \pi^{-1}(\eta))$ has an embedding into the sphere bundle of ν given by the perturbation. Furthermore, $q_\sim(\Sigma_{(1,(k+1)/2-1}) \cap \pi^{-1}(\eta))$ is a double covering of $\Sigma_{(k+1)/2}^{k+1,0}$ given by this embedding, and it is immersed with a trivial normal bundle $\tilde{\nu}$ into the tubular neighborhood of $f(\Sigma_{(k+1)/2}^{k+1,0})$. Moreover there is a natural trivialization of $\tilde{\nu}$ corresponding to the indices of the singular set of $\Theta|_{\pi^{-1}(p)}$, $p \in \Sigma_{(k+1)/2}^{k+1,0}$. Thus the perturbation Θ satisfies the requirements of the theorem. Hence Θ is a tame corank 1 map and applying Proposition 3.7 finishes the proof. \square

Remark 4.1. The double covering of $\Sigma_{(k+1)/2}^{k+1,0}$ defined by $\Theta(\Sigma_{(1,(k+1)/2-1}) \cap \pi^{-1}(\eta))$ is trivial if and only if ν is trivial.

Remark 4.2. If $k = 1$, then Θ is obviously a fold map. For example, for a Morse function $f: S^2 \rightarrow \mathbb{R}$ with three definite and one indefinite critical points, $\text{Bl}_\eta S^2 = S^2 \# \mathbb{R}P^2 = \mathbb{R}P^2$ and Θ is a Morse function with three definite and two indefinite critical points. It can be seen that Θ has two singular fibers containing indefinite critical points and exactly one of them has non-orientable neighborhood.

Lemma 4.3. *Let X be a manifold and l be a line bundle over X . Assume that there is an open covering $X_0 \cup X_1 = X$ such that the bundle l is trivial over both X_0 and X_1 . Then there exists an epimorphism $\varepsilon_X^2 \rightarrow l$.*

Proof. Let $f_i: l|_{X_i} \rightarrow \mathbb{R}$ be fiberwise linear isomorphisms which trivialize l over X_i for $i = 0, 1$. Since X_0 and X_1 are open, we can choose continuous functions $\lambda_0, \lambda_1: X \rightarrow [0, 1]$ such that $\lambda_i^{-1}(0) = X - X_i$ for $i = 0, 1$. Define the fiberwise linear function $f: l \rightarrow \mathbb{R}^2$ by the formula

$$f(v) = (\lambda_0(x)f_0(v), \lambda_1(x)f_1(v))$$

for each vector $v \in l$ over the point $x \in X$. This definition makes sense since $\lambda_i(x) = 0$ whenever $f_i(x)$ is not defined. Over X_i , the map f composed with the projection onto the i -th coordinate is a nonzero rescaling of f_i , $i = 0, 1$, thus f is injective and we can identify l with a subbundle of ε_X^2 . Taking a Riemannian metric on ε_X^2 allows us to consider the orthogonal projection $\varepsilon_X^2 \rightarrow l$, which is an epimorphism. \square

Proof of Theorem 2.6. Suppose $f: M^n \rightarrow Q^{n-k}$ is a Morin map. By [Ch83] the cokernel bundle of $df: TM \rightarrow f^*TQ$ is trivial over a neighborhood of $\Sigma^{k+1,1}$ in Σ . Note that coker df has natural orientations over the fold singularities of index not equal to $(k+1)/2$, and these orientations agree when two fold singular sets are attached to each other along $\Sigma^{k+1,1}$. Let $\Sigma_{\geq 0}$ denote the complement of a small regular neighborhood of $\Sigma^{k+1,1} \cup \bigcup_{\lambda \neq (k+1)/2} \Sigma_\lambda^{k+1,0}$ in Σ . Then coker df is trivial on $\Sigma - \Sigma_{\geq 0}$. If $\Sigma_{\geq 0} = \emptyset$, then f is a tame corank 1 map and by Proposition 3.7 we obtain the statement of the theorem. Assume that $\Sigma_{\geq 0} \neq \emptyset$.

Take the blowup $\text{Bl}_\xi f: \text{Bl}_\xi M \rightarrow Q$. Define a smooth non-negative function Δ on $\Sigma_{\geq 0}$ such that it vanishes on $\Sigma_{=0}$, where $\Sigma_{=0}$ is a small compact regular neighborhood of $\partial\Sigma_{\geq 0}$, and takes small positive values on $\Sigma_{>0}$, where $\Sigma_{>0} = \Sigma_{\geq 0} - \Sigma_{=0}$. Then, perturb the map $\text{Bl}_\xi f: \text{Bl}_\xi M \rightarrow Q$ in $\pi^{-1}(\eta|_{\Sigma_{\geq 0}})$ in the same way as we did in Theorem 2.5 but replace ε in (4.1) by the value of the function Δ . Denote the resulting map by $\widetilde{\text{Bl}_\xi f}$. Note that on $\text{Bl}_\xi M - \pi^{-1}(\eta|_{\Sigma - \Sigma_{>0}})$ the restrictions of $\widetilde{\text{Bl}_\xi f}$ and $\text{Bl}_\xi f$ coincide. Denote by $\tilde{\Sigma}_{>0}$ the singular set of $\widetilde{\text{Bl}_\xi f}|_{\pi^{-1}(\eta|_{\Sigma_{>0}})}$. The singular set of $\widetilde{\text{Bl}_\xi f}|_{\pi^{-1}(\xi|_{\Sigma - \Sigma_{>0}})}$ is equal to $\pi^{-1}(\Sigma - \Sigma_{>0})$.

Since $d\text{Bl}_\xi f(T\text{Bl}_\xi M) = df d\pi(T\text{Bl}_\xi M) = df(TM)$ and $f^*TQ/f^*df(TM)$ is trivial on $\Sigma - \Sigma_{>0}$, clearly $\text{coker } d\text{Bl}_\xi f = \pi^*f^*(TQ)/\pi^*f^*d\text{Bl}_\xi f(T\text{Bl}_\xi M)$ is also trivial on $\pi^{-1}(\Sigma - \Sigma_{>0})$. Hence so is $\text{coker } d\widetilde{\text{Bl}_\xi f}$. Moreover $d\widetilde{\text{Bl}_\xi f}$ also has trivial cokernel on $\tilde{\Sigma}_{>0}$ by Theorem 2.5.

Apply Lemma 4.3 to $l = \text{coker } d\widetilde{\text{Bl}_\xi f}$ over $X = \tilde{\Sigma}_{>0} \cup \pi^{-1}(\Sigma - \Sigma_{>0})$ with the covering consisting of $X_0 = \tilde{\Sigma}_{>0}$ and X_1 being a small neighborhood of $\pi^{-1}(\Sigma - \Sigma_{>0})$. The argument above ensures that l is indeed trivial when restricted to either X_0 or X_1 since X_1 is a deformation retract of $\pi^{-1}(\Sigma - \Sigma_{>0})$. Therefore there exists a fiberwise epimorphism $\sigma: \varepsilon_X^2 \rightarrow \text{coker } d\widetilde{\text{Bl}_\xi f}|_X$. Compose σ with the standard embedding $\text{coker } d\widetilde{\text{Bl}_\xi f}|_X \rightarrow (\widetilde{\text{Bl}_\xi f})^*TQ|_X$ and then extend this composite map to all of $\text{Bl}_\xi M$ as a linear bundle map $\tilde{\sigma}: \varepsilon_{\text{Bl}_\xi M}^2 \rightarrow (\widetilde{\text{Bl}_\xi f})^*TQ$ by scaling it with a bump function concentrated on a small neighborhood of X . Combining $d\widetilde{\text{Bl}_\xi f}$ with $\tilde{\sigma}$ we get a bundle map

$$d\widetilde{\text{Bl}_\xi f} + \tilde{\sigma}: T\text{Bl}_\xi M \oplus \varepsilon^2 \rightarrow (\widetilde{\text{Bl}_\xi f})^*TQ$$

which is obviously surjective both on $\text{Bl}_\xi M - X$ and X . This completes the proof. \square

5. COMPUTING THE CHARACTERISTIC CLASSES OF THE SOURCE MANIFOLD

Let γ denote the line bundle over Σ defined by the condition that $w_1(\gamma)$ is Poincaré dual to the class represented by $\Sigma^{k+1,1}$. We relate $f^*df(TM)|_\Sigma$ to $T\Sigma$ by the following

Proposition 5.1. *For a cusp map f , we have $T\Sigma \oplus \gamma \cong f^*df(TM)|_\Sigma \oplus \varepsilon^1$.*

Proof. Denote by C the manifold $\Sigma^{k+1,1}$. Since f is a cusp map, we have $C = \Sigma^{k+1,1,0}$. We will first construct a bundle monomorphism

$$i: T\Sigma \rightarrow f^*df(TM)|_\Sigma \oplus \varepsilon^1$$

covering the identity map of Σ . Apart from C , the map df is an isomorphism between $T\Sigma$ and $f^*df(TM)|_\Sigma$. On C the restriction of df to TC is a monomorphism hence there is an isomorphism

$$j: T\Sigma|_C \rightarrow f^*df(TC)|_C \oplus \ker d(f|_\Sigma)|_C$$

defined by composing the isomorphism $T\Sigma|_C \cong TC \oplus \ker d(f|_\Sigma)|_C$ with the isomorphism

$$(df|_{TC}, \text{id}_{\ker d(f|_\Sigma)|_C}): TC \oplus \ker d(f|_\Sigma)|_C \rightarrow f^*df(TC)|_C \oplus \ker d(f|_\Sigma)|_C.$$

Denote by $\text{pr}_{\ker}: T\Sigma|_C \rightarrow \ker d(f|_\Sigma)|_C$ the composition of j with the projection of $f^*df(TC) \oplus \ker d(f|_\Sigma)|_C$ to the second factor.

Since $\ker d(f|_\Sigma)|_C \subset T\Sigma$ is never tangent to C , we can identify it with the normal bundle of C in Σ . But k is odd hence this normal bundle is trivial – the indices of fold points on the two sides are different. After choosing a trivialization of $\ker d(f|_\Sigma)|_C$, the map j can be considered as an embedding of $T\Sigma|_C$ into $f^*df(TM)|_C \oplus \varepsilon^1$, with its image $\text{im } j = f^*df(TC)|_C \oplus \varepsilon^1$. This embedding extends as a fiberwise embedding onto a small neighborhood $N(C)$ of C in Σ , and we will consider j and also pr_{\ker} to be defined on $N(C)$.

Define i to be the linear interpolation of the embedding $j: T\Sigma|_{N(C)} \rightarrow f^*df(TM)|_{N(C)} \oplus \varepsilon^1$ and the map $(df|_{T\Sigma}, 0): T\Sigma \rightarrow f^*df(TM)|_\Sigma \oplus \varepsilon^1$. That is, we take a bump function $\lambda: T\Sigma \rightarrow [0, 1]$ such that $\lambda = 0$ outside over a tubular neighborhood of C in Σ and $\lambda^{-1}(\{1\}) = T\Sigma|_C$, and we define i to be $(df|_{T\Sigma}, \lambda \text{pr}_{\ker}): T\Sigma \rightarrow f^*df(TM)|_\Sigma \oplus \varepsilon^1$. Thus i is well-defined, since $\lambda = 0$ where pr_{\ker} is not defined, and it is clear that i has full rank both on C and its complement in Σ .

From this embedding i , we get $f^*df(TM)|_\Sigma \oplus \varepsilon^1 \cong T\Sigma \oplus \text{coker } i$, and we only need to identify $\text{coker } i$ with γ . Indeed, on the set $\Sigma - C$ the line bundle $\text{coker } i$ is trivial as $\text{im } i$ projects isomorphically onto $f^*df(TM)|_\Sigma$. On a tubular neighborhood of C , this trivialization of $\text{coker } i|_{\Sigma-C}$ has the opposite signs on the two sides of C , thus $w_1(\text{coker } i)$ is dual to C in Σ as claimed. \square

Corollary 5.2. *For a cusp map f , we have $w(T\Sigma) = w(f|_\Sigma^*TQ)w(\nu)^{-1}w(\gamma)^{-1}$, where ν denotes the line bundle $f^*TQ|_\Sigma/f^*df(TM)|_\Sigma$.*

Let $c \in H^1(\Sigma)$ denote the characteristic class $w_1(\gamma)$. As noted above, the manifold $\Sigma^{k+1,1}$ has a trivial normal bundle in Σ , thus $c^2 = 0$. Let b denote $w_1(f^*TQ|_\Sigma/f^*df(TM)|_\Sigma) \in H^1(\Sigma)$.

Proof of Theorem 3.1. Let δ be 0 if f is a fold map, and let δ be 1 otherwise. If the Morin map $f: M^n \rightarrow Q^{n-k}$ is not a cusp map and both M and Q are orientable, then perturb f to get a cusp map, see [Sad03], and denote this cusp map by f as well for simplicity.

By the blowup formula for Stiefel-Whitney classes [GP07, Theorem 10 and Remark (2) on page 328], we can express in our notation the Stiefel-Whitney classes of TM in terms of the classes of $T\text{Bl}_\xi M$, $T\Sigma$ and ξ in the following way:

$$\begin{aligned} w(T\text{Bl}_\xi M) - w(\pi^*TM) &= \\ &= \tilde{w}_1 \left(w(p^*T\Sigma) \frac{1}{w_1(\mu)} \left(\sum_{t=0}^{k+1} w_t(p^*\xi)(1 + w_1(\mu))^{k+2-t} - w(p^*\xi) \right) \right). \end{aligned}$$

Here μ denotes the canonical line bundle over $\pi^{-1}(\Sigma)$. Recall that p is the restriction $\pi|_{\pi^{-1}(\Sigma)}$, and the map $\tilde{i}: \pi^{-1}(\Sigma) \rightarrow \text{Bl}_\xi M$ is the natural embedding. By Theorem 2.6 and Remark 2.8, the total Stiefel-Whitney class of $T \text{Bl}_\xi M$ is equal to the product of the total Stiefel-Whitney class of a $(k+1+\delta)$ -dimensional bundle and the total Stiefel-Whitney class of $\pi^* f^* TQ$, which contains no term of degree greater than K . Therefore $w_l(T \text{Bl}_\xi M) = 0$ for $l > k + K + 1 + \delta$.

Expanding the blowup formula for $r \geq k + K + 2 + \delta$ we thus get

$$\begin{aligned} w_r(\pi^* TM) &= w_r(T \text{Bl}_\xi M) + \\ &+ \tilde{i}_! \left(\sum_{q=0}^{r-1} w_{r-1-q}(p^* T\Sigma) \left[\sum_{t=0}^{k+1} w_t(p^* \xi) \sum_{s=0}^{k+1-t} \binom{k+2-t}{s+1} w_1(\mu)^s \right]_{\deg=q} \right) = \\ &= \tilde{i}_! \left(\sum_{q=0}^{k+1} p^* w_{r-1-q}(T\Sigma) \sum_{t=0}^{k+1} \binom{k+2-t}{q-t+1} p^* w_t(\xi) w_1(\mu)^{q-t} \right), \end{aligned}$$

where we use our convention about binomial coefficients and $w_1(\mu)^{-1}$ is defined to be 0.

The classes $w_{r-1-q}(T\Sigma)$ can be obtained from Corollary 5.2. Under the assumption that $m \geq K + \delta$, we have

$$\begin{aligned} w_m(T\Sigma) &= [w(f|_\Sigma^* TQ) w(\nu)^{-1} w(\gamma)^{-1}]_{\deg=m} = \\ &= \sum_{l=0}^m f|_\Sigma^* w_l(TQ) (b^{m-l} + b^{m-l-1} c) = \sum_{l=0}^K f|_\Sigma^* w_l(TQ) (b^{m-l} + b^{m-l-1} c). \end{aligned}$$

Notice that in the formula all the exponents of b are at least 0. Substituting $m = r - 1 - q$, where $0 \leq q \leq k + 1$, we get that

$$w_{r-1-q}(T\Sigma) = b^{r-1-q-K-\delta} w_{K+\delta}(T\Sigma)$$

for all $0 \leq q \leq k + 1$. Hence we have that if $r \geq k + K + 2 + \delta$, then

$$\begin{aligned} w_r(\pi^* TM) &= \\ &= \tilde{i}_! \left(\sum_{q=0}^{k+1} p^* \left(b^{r-1-q-K-\delta} w_{K+\delta}(T\Sigma) \right) \sum_{t=0}^{k+1} \binom{k+2-t}{q-t+1} p^* w_t(\xi) w_1(\mu)^{q-t} \right) = \\ &= \tilde{i}_! \left(p^* b^{r-k-K-2-\delta} \sum_{q=0}^{k+1} \sum_{t=0}^{k+1} \binom{k+2-t}{q-t+1} p^* \left(b^{k+1-q} w_{K+\delta}(T\Sigma) w_t(\xi) \right) w_1(\mu)^{q-t} \right). \end{aligned}$$

Notice that the double sum in this formula does not depend on r at all, and let α denote

$$\sum_{q=0}^{k+1} \sum_{t=0}^{k+1} \binom{k+2-t}{q-t+1} p^* \left(b^{k+1-q} w_{K+\delta}(T\Sigma) w_t(\xi) \right) w_1(\mu)^{q-t}.$$

Then

$$w_r(\pi^* TM) = \tilde{i}_! \left(\alpha p^* b^{r-k-K-2-\delta} \right)$$

holds for all $r \geq k + K + 2 + \delta$ and we can calculate products of these characteristic classes by repeatedly applying the formula $\tilde{i}_!(u) \tilde{i}_!(u) = \tilde{i}_!(1) \tilde{i}_!(uv)$ as follows. For $r_1, \dots, r_m \geq k + K +$

$2 + \delta$, we have

$$\begin{aligned} w_{r_1}(\pi^*TM) \dots w_{r_m}(\pi^*TM) &= \prod_{i=1}^m \tilde{t}_i \left(\alpha p^* b^{r_i - k - K - 2 - \delta} \right) = \\ &= \tilde{t}_1(1)^{m-1} \tilde{t}_1 \left(\prod_{i=1}^m \alpha p^* b^{r_i - k - K - 2 - \delta} \right) = \tilde{t}_1(1)^{m-1} \tilde{t}_1 \left(\alpha^m p^* b^{\sum_{i=1}^m (r_i - k - K - 2 - \delta)} \right). \end{aligned}$$

This expression clearly depends only on m and the sum $r_1 + \dots + r_m$, and since the homomorphism $\pi^*: H^*(M; \mathbb{Z}_2) \rightarrow H^*(\text{Bl}_\xi M; \mathbb{Z}_2)$ is injective, this proves the statement of the theorem. \square

Lemma 5.3. *Let $n = 2^D + m$, $0 \leq m < 2^D$. Then $\binom{n}{m}$ is odd, and $\binom{n}{r}$ is even for all r satisfying $m < r < 2^D$.*

Proof. A criterion of [Gl99] states that $\binom{b}{a}$, $0 \leq a \leq b$, is even if and only if there is a binary position at which a has the digit 1 and b has the digit 0. This criterion shows that $\binom{n}{m}$ is odd. If $\binom{n}{r}$ is odd for some $0 \leq r < 2^D$, then all the binary digits of r at the positions where n has 0 have to be 0 as well. Since r has binary length at most D , this is equivalent to the condition that r has binary digits 1 only at positions where $m = n - 2^D$ has 1 as well, hence the maximal such r is m as claimed. \square

Proposition 5.4. *Let $n = 2^D + m$ with $0 \leq m < 2^D - 2$. Assume there exists an integer l such that the equations $w_a w_b(\mathbb{R}P^n) = w_c w_d(\mathbb{R}P^n)$ hold for all $a, b, c, d \geq l$. Then $l \geq m + 1$.*

Proof. Denote the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ by x , then we have

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1})$$

and

$$w_j(\mathbb{R}P^n) = \binom{n+1}{j} x^j.$$

By Lemma 5.3 $\binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1}$ is odd, hence the class $w_{m+1}(\mathbb{R}P^n)$ is the generator x^{m+1} , while the classes $w_{m+2}(\mathbb{R}P^n), \dots, w_{2^D-1}(\mathbb{R}P^n)$ vanish. Note that there is at least one class in this latter list due to the constraint $m < 2^D - 2$. In particular, the class $w_m w_{m+2}(\mathbb{R}P^n)$ also has to vanish, while $w_{m+1}^2(\mathbb{R}P^n) = x^{2m+2}$ is not zero as $2m+2 < 2^D - 2 + m + 2 = n < n+1$. Therefore the relation $w_{m+1}^2 = w_{m+2} w_m$ does not hold on $\mathbb{R}P^n$, implying $l > m$. \square

Proposition 5.5. *Let $n = 2^D + m$ with $0 \leq m < 2^D - 2$. Then the relations*

$$\prod_{i \in I} w_i(\mathbb{R}P^n) = \prod_{j \in J} w_j(\mathbb{R}P^n)$$

hold for all $I, J \subseteq \{0, \dots, n\}$ which satisfy $|I| = |J|$, $\min I, \min J \geq m + 1$ and $\sum_{i \in I} i = \sum_{j \in J} j$.

Proof. As before, we note that the classes $w_{m+2}(\mathbb{R}P^n), \dots, w_{2^D-1}(\mathbb{R}P^n)$ vanish.

For $|I| = 1$ the statement is trivial. For $|I| \geq 2$ such that $\min I \geq m + 1$ we have three possibilities:

- I consists of a number of copies of $m + 1$. Then the only J which satisfies both $\min J \geq m + 1$ and $\sum_{j \in J} j = \sum_{i \in I} i = (m + 1)|I|$ is I itself.
- I contains an index between $m + 2$ and $2^D - 1$. Then $\prod_{i \in I} w_i(\mathbb{R}P^n)$ contains a zero class and thus vanishes.

- I contains at least one index greater than $2^D - 1$. Then taking any other index $j \in I$ we have $\sum_{i \in I} i \geq 2^D + j \geq 2^D + m + 1 = n + 1$. Therefore $\prod_{i \in I} w_i(\mathbb{R}P^n)$ has degree greater than n and consequently vanishes.

Observe that for any J satisfying the requirements of the proposition we have the analogous three possibilities, hence any such J gives the same product of Stiefel-Whitney classes as I . \square

Remark 5.6. In the cases $n = 2^D - 2$ and $n = 2^D - 1$ the nontrivial characteristic classes of $\mathbb{R}P^n$ are either all the generators of the respective cohomology groups $H^*(\mathbb{R}P^n)$ or all vanish, therefore our multiplicativity condition is satisfied for all indices.

Proof of Theorem 3.5. Equip TM and TQ with Riemannian metrics, thus identifying sections of these bundles with 1-forms. Assume that we have a trivialization of $TQ \oplus \varepsilon^l$ given by a collection of $n - k + l$ linearly independent 1-forms. Then any smooth map $f: M \rightarrow Q$ defines pullbacks of these 1-forms to $TM \oplus \varepsilon^l$ via df . By the assumption of the theorem, $\text{rank } df \geq n - k - 1$ at all points of M , thus the linear span of the pulled-back forms is at least $n - k + l - 1$. The metric on TM identifies these forms with $n - k + l$ vector fields which have a linear span of dimension at least $n - k + l - 1$ everywhere. By [Ga78, Po47, Ro52], the rational Pontryagin class $p_i^{\mathbb{Q}}(TM) = p_i^{\mathbb{Q}}(TM \oplus \varepsilon^l)$ is represented by the locus where $n + l - 2i + 2$ generic sections of $TM \oplus \varepsilon^l$ lie in a subspace of dimension at most $n + l - 2i$. This class therefore vanishes if $n - k + l \geq n + l - 2i + 2$, that is, when $2i \geq k + 2$. \square

Proof of Proposition 3.7. (2) \implies (1): By [An04], if there is a $TM \oplus \varepsilon^1 \rightarrow TQ$ epimorphism, then there is a fold map $M \rightarrow Q$ with orientable singular set. (1) \implies (2): Assume that we have a tame corank 1 map $f: M \rightarrow Q$. The bundle $\text{coker } df|_{\Sigma} = (f^*TQ/f^*df(TM))|_{\Sigma}$ is considered as a subbundle of f^*TQ and it is trivial. Similarly to [An04, Proof of Lemma 3.1], let $L: \varepsilon^1 \rightarrow TQ$ be an extension of the bundle monomorphism $\text{coker } df|_{\Sigma} \rightarrow f^*TQ \rightarrow TQ$ as a bundle homomorphism covering f . Then $df + L$ is an epimorphism $TM \oplus \varepsilon^1 \rightarrow TQ$.

Finally, if (1) or (2) holds and Q is stably parallelizable, then by the above, we have $TM \oplus \varepsilon^1 \oplus \varepsilon^N \cong \zeta \oplus f^*TQ \oplus \varepsilon^N \cong \zeta \oplus \varepsilon^{N+n-k}$ for some $N \gg 0$ and a $(k+1)$ -dimensional bundle ζ . Thus $g.\dim([TM] - [\varepsilon^n]) \leq k + 1$.

If Q is stably parallelizable and $g.\dim([TM] - [\varepsilon^n]) \leq k + 1$, then $TM \oplus \varepsilon^N \cong \zeta^{k+1} \oplus \varepsilon^{N+n-k-1} \cong \zeta^{k+1} \oplus TQ \oplus \varepsilon^{N-1}$ for some $N \gg 0$, and thus $TM \oplus \varepsilon^1 \cong \zeta^{k+1} \oplus TQ$, which proves (2). \square

Remark 5.7. If there is a tame Morse-Bott map $f: P^n \rightarrow Q^{n-k}$, then $TP \oplus \varepsilon^1$ splits as $\zeta^{k+1} \oplus f^*TQ$ for some $(k+1)$ -dimensional vector bundle ζ^{k+1} .

Proof of Proposition 3.10. Let $\varphi(n)$ denote the cardinality of the set $\{0 < s \leq n : s \equiv 0, 1, 2, 4 \pmod{8}\}$. By [At61, §5], $[T\mathbb{R}P^n] - [\varepsilon^n] = (n+1)x$ and $\gamma^i([T\mathbb{R}P^n] - [\varepsilon^n]) = 2^{i-1} \binom{n+1}{i} x$, $i \geq 1$, where x denotes the generator of $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n) = \mathbb{Z}_{2^{\varphi(n)}}$. Therefore $\gamma^i([T\mathbb{R}P^n] - [\varepsilon^n]) = 0$ if and only if $2^{\varphi(n)}$ divides $2^{i-1} \binom{n+1}{i}$. Let $r(n)$ denote the greatest integer s for which $2^{s-1} \binom{n+1}{s}$ is not divisible by $2^{\varphi(n)}$. Then by Proposition 3.7 there is no tame corank 1 map of $\mathbb{R}P^{2^n-1}$ into $\mathbb{R}P^{2^n-1-k}$ for $k \leq r(2^n-1) - 2$. It is easy to see that $\varphi(2^n-1) = 2^{n-1} - 1$ if $n \geq 3$. By a classical result of E. Kummer, the highest power $c(s)$ of 2 which divides $\binom{2^n}{s}$ can be obtained by counting the number of carries when s and $2^n - s$ are added in base 2. For $s \leq 2^{n-1} - 1$, we claim that $c(s) = n - R(s)$, where $2^{R(s)}$ is the maximal power of 2 which divides s . Indeed, $2^n - 1 - s$ is obtained by negating the binary form of s bitwise, hence $2^n - s$ is obtained by negating the binary form of s bitwise from the $(n-1)$ st to the $R(s)$ th binary position, where both of s and $2^n - s$ have the digit 1, and after that position both have digits 0. Therefore

when we add s and $2^n - s$ in base 2, we have $n - R(s)$ carries. By the definition of $r(n)$ it follows that $r(2^n - 1)$ is the largest integer s for which $s + n - R(s) < 2^{n-1}$. \square

5.1. Computing the cobordism class of the source manifold. Theorem 3.1 gives us relations among the characteristic numbers of a source manifold of a Morin map as well. However, by following a different line of argument, we can obtain more relations among the characteristic numbers as follows.

For a Morin map $f: M^n \rightarrow Q^{n-k}$ with odd $k \geq 1$, let us denote by N_Σ the projectivization $\mathbb{R}P(\xi \oplus \varepsilon^1)$ of the $(k+2)$ -dimensional vector bundle $\xi \oplus \varepsilon^1$ over the singular set Σ , where ξ denotes the normal bundle of Σ . Thus N_Σ is a closed n -dimensional manifold fibered over Σ with $\mathbb{R}P^{k+1}$ as fiber. Let $\tau: N_\Sigma \rightarrow \Sigma$ denote this fibration.

Lemma 5.8. *The blowup $\text{Bl}_\xi M$ is cobordant to the disjoint union of M and N_Σ .*

Proof. Consider the disk bundle $D(\xi \oplus \varepsilon^1)$ of $\xi \oplus \varepsilon^1$. Let U and V be small neighborhoods of $\xi \oplus \{1\}$ and $\xi \oplus \{-1\}$ respectively in the boundary $\partial D(\xi \oplus \varepsilon^1)$. The total space of $D(\xi \oplus \varepsilon^1)$ can be naturally glued to the boundary component $(M \sqcup \mathbb{R}P(\xi \oplus \varepsilon^1)) \times \{0\}$ of $(M \sqcup \mathbb{R}P(\xi \oplus \varepsilon^1)) \times [0, 1]$ along U identified with the total space of ξ as an open submanifold of M and along V identified with $[\xi : 1] \subset \mathbb{R}P(\xi \oplus \varepsilon^1)$. After smoothing the corners introduced by the gluing, the resulting $(n+1)$ -manifold has boundary consisting of the disjoint union of M , $\mathbb{R}P(\xi \oplus \varepsilon^1)$ and $\text{Bl}_\xi M$. This completes the proof. \square

Remark 5.9. As one can see easily, the cobordism in Lemma 5.8 extends naturally to a bordism of the maps $\text{Bl}_\xi f: \text{Bl}_\xi M \rightarrow Q$ and the union $f \sqcup f|_\Sigma \circ \tau: M \sqcup N_\Sigma \rightarrow Q$. Indeed, we can map all points of each fiber of $D(\xi \oplus \varepsilon^1)$ over p to p , where $p \in \Sigma$, and then into Q by $f|_\Sigma$. Thus the evaluation of any “characteristic number” w_I (i.e. degree n monomial of Stiefel-Whitney characteristic classes) of $T\text{Bl}_\xi M - (\text{Bl}_\xi f)^*TQ$ on the fundamental class $[\text{Bl}_\xi M] \in H_n(\text{Bl}_\xi M; \mathbb{Z}_2)$ is equal to the sum of the evaluations of $w_I(TM - f^*TQ)$ and $w_I(TN_\Sigma - \tau^*f|_\Sigma^*TQ)$ on the fundamental classes $[M]$ and $[N_\Sigma]$, respectively.

Recall that b denotes $w_1(f^*TQ|_\Sigma / f^*df(TM)|_\Sigma) \in H^1(\Sigma)$ and $w_1(\gamma) = c \in H^1(\Sigma)$ is the Poincaré dual to the class represented by $\Sigma^{k+1,1}$. We have also seen that $c^2 = 0$.

Proposition 5.10. *Let $f: M^n \rightarrow Q^{n-k}$ be a cusp map. Let δ be 0 if f is a fold map, and let δ be 1 otherwise.*

(1) *For $r \geq k+1+\delta$, the degree r term of $w(TN_\Sigma - \tau^*f|_\Sigma^*TQ)$ has the form*

$$\tau^*b^{r-k-1-\delta}w_{k+1+\delta}(TN_\Sigma - \tau^*f|_\Sigma^*TQ).$$

(2) *Any two characteristic numbers of $w(TN_\Sigma - \tau^*f|_\Sigma^*TQ)$ which contain the same number of multiplicands and contain no instances of $w_1, \dots, w_{k+\delta}$ are equal.*

(3) *For any multiindex $J = (j_1, \dots, j_l)$ such that $\sum_{i=1}^l j_i = n$ and $j_i \geq k+2+\delta$ for some $1 \leq i \leq l$, the characteristic numbers $\langle w_J(TM - f^*TQ), [M] \rangle$ and $\langle w_J(TN_\Sigma - \tau^*f|_\Sigma^*TQ), [N_\Sigma] \rangle$ coincide. The characteristic numbers defined by $w(TM - f^*TQ)$ which involve no $w_1, \dots, w_{k+\delta}$ satisfy the property of depending only on the number of multiplicands.*

Proof. The fibration $\tau: N_\Sigma \rightarrow \Sigma$ has fiber $\mathbb{R}P^{k+1}$ and TN_Σ splits into the direct sum of the horizontal component $\tau^*T\Sigma$ and the vertical component ψ having rank $k+1$, which is tangent to the fibers. By Corollary 5.2, we have

$$w(f|_\Sigma^*TQ)^{-1}w(T\Sigma) = w(\nu)^{-1}w(\gamma)^{-1} = (1+b)^{-1}(1+c)^{-1} = \sum_{i=0}^{\infty} b^i(1+c).$$

Hence we can express $w(\tau^*f|_{\Sigma}^*TQ)^{-1}w(TN_{\Sigma})$ as

$$\begin{aligned} w(\tau^*f|_{\Sigma}^*TQ)^{-1}w(TN_{\Sigma}) &= w(\tau^*f|_{\Sigma}^*TQ)^{-1}\tau^*w(T\Sigma)w(\psi) = \\ &= \left(1 + \sum_{i=1}^{\infty}(\tau^*b^i + \tau^*b^{i-1}c)\right) \left(\sum_{j=0}^{k+1}w_j(\psi)\right) = \\ &\stackrel{(a)}{=} \sum_{r=0}^{\infty} \sum_{j=0}^{\min\{r,k+1\}} w_j(\psi)(\tau^*b^{r-j} + \tau^*b^{r-j-1}c), \end{aligned}$$

where (a) follows from rearranging the sums by $i + j = r$, and we use the convention that $b^{-1} = 0$. The class $w_r(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ)$ in the case of $r \geq k + 2$ therefore has the form

$$\begin{aligned} w_r(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ) &= \sum_{j=0}^{k+1} w_j(\psi)(\tau^*b^{r-j} + \tau^*b^{r-j-1}c) = \\ &= \tau^*b^{r-k-2}w_{k+2}(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ). \end{aligned}$$

If additionally $\delta = 0$ (thus $c = 0$), then similarly we have

$$w_r(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ) = \sum_{j=0}^{k+1} w_j(\psi)\tau^*b^{r-j} = \tau^*b^{r-k-1}w_{k+1}(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ)$$

for $r \geq k + 1$. These two equalities prove (1).

Consider now a product $\prod_{i=1}^m w_{j_i}(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ)$ which contains no instances of $w_1(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ), \dots, w_{k+\delta}(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ)$. By (1), it has the form

$$\tau^*b^{n-(k+1+\delta)m}w_{k+1+\delta}^m(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ).$$

In particular, this expression depends only on m and hence any two characteristic numbers of $w(TN_{\Sigma} - \tau^*f|_{\Sigma}^*TQ)$ with the same number of multiplicands and no instances of $w_1, \dots, w_{k+\delta}$ are equal. This proves (2).

By Theorem 2.6 and Remark 2.8 the formal difference bundle $T\text{Bl}_{\xi}M - \pi^*f^*TQ$ is stably equivalent to a $(k + 1 + \delta)$ -dimensional bundle. Therefore the characteristic classes $w_r(T\text{Bl}_{\xi}M - \pi^*f^*TQ)$ of $T\text{Bl}_{\xi}M - \pi^*f^*TQ$ with $r > k + 1 + \delta$ vanish. Thus, by Remark 5.9 those characteristic numbers of the virtual normal bundles of the maps $N_{\Sigma} \rightarrow Q$ and $M \rightarrow Q$ which contain w_r with $r > k + 1 + \delta$ coincide. This finishes the proof of (3). \square

Corollary 5.11. *Let $w_1(TM), \dots, w_k(TM) = 0$ and Q be stably parallelizable. If there exists a fold map $M^n \rightarrow Q^{n-k}$, then each of the nonzero characteristic numbers of M , which has more than one multiplicand, is equal to a number of the form $w_{k+1}^l w_{n-(k+1)l}[M]$ with $0 \leq l \leq \frac{n}{k+1} - 1$.*

5.1.1. *Adding Dold relations to the relations of Proposition 5.10.* In this section, we work in the case of fixed $n \geq 2$, $k = 1$ and assume an orientable source manifold M^n .

Denote by \mathcal{I} the linear space in the graded \mathbb{Z}_2 -algebra $\mathbb{Z}_2[w_1, \dots, w_n, \dots]$ spanned by the set

$$\begin{aligned} \{q_1 - q_2 \in \mathbb{Z}_2[w_1, \dots, w_n]_{\deg=n} : q_1, q_2 \text{ are monomials in } \mathbb{Z}_2[w_2, \dots, w_n]_{\deg=n}, \\ |q_1| = |q_2|\} \cup \{q_1 \in [w_1\mathbb{Z}_2[w_1, \dots, w_n]]_{\deg=n}\}, \end{aligned}$$

where $|q|$ denotes the length of the monomial q , i.e. the number of (not necessarily different) indeterminants whose product is q . Proposition 5.10 (3) states that if M is an n -manifold admitting a codimension -1 fold map into a stably parallelizable target, then the evaluation $w_i = w_i(TM)$ sends all the members of \mathcal{I} to 0.

Denote by \mathcal{D} the linear space spanned in $\mathbb{Z}_2[w_1, \dots, w_n, \dots]$ by the set

$$\{q \in \mathbb{Z}_2[w_1, \dots, w_n] : p \in \mathbb{Z}_2[w_1, \dots, w_n, \dots]_{\deg \leq n-1}, q = [w^{-1} \cdot \text{Sq } p]_{\deg=n}\},$$

where w stands for the total Stiefel-Whitney class $1 + w_1 + \dots + w_n + \dots$. We will apply a result of Dold [Do56] which states that all the relations between the characteristic numbers of n -manifolds are exactly those of the form $q = 0$ for $q \in \mathcal{D}$. Combining this set of relations with $\{q = 0 : q \in \mathcal{I}\}$ and proving that $\dim \mathbb{Z}_2[w_1, \dots, w_n]_{\deg=n} / (\mathcal{D} \oplus \mathcal{I}) = 0$ unless $n = 2^a + 2^b - 1$ for some $a > b \geq 0$ forms the core of the proof of Theorem 3.12.

To utilize the relations obtained in Proposition 5.10 (3), we will consider the graded algebra homomorphism

$$\varrho: \mathbb{Z}_2[w_1, \dots, w_n, \dots] \rightarrow \mathbb{Z}_2[x, t], \quad \deg x = 2, \deg t = 1$$

$$\varrho(w_1) = 0, \varrho(w_s) = xt^{s-2} \text{ for } s \geq 2.$$

Define $\text{im } \varrho_n$ to be $\mathbb{Z}_2[x, t]_{\deg=n} \cap \text{im } \varrho = \langle xt^{n-2}, \dots, x^{[n/2]} t^{n-2[n/2]} \rangle$. It is straightforward to see that $\ker \varrho \cap \mathbb{Z}_2[w_1, \dots]_{\deg=n}$ is exactly \mathcal{I} , therefore

$$\begin{aligned} \dim \mathbb{Z}_2[w_1, \dots]_{\deg=n} / (\mathcal{D} \oplus \mathcal{I}) &= \dim (\mathbb{Z}_2[w_1, \dots]_{\deg=n} / \mathcal{I}) / (\mathcal{D} / \mathcal{D} \cap \mathcal{I}) = \\ &= \dim (\mathbb{Z}_2[w_1, \dots]_{\deg=n} / \mathcal{I}) - \dim \text{im } \varrho|_{\mathcal{D}} = \dim \text{im } \varrho_n - \dim \text{im } \varrho|_{\mathcal{D}}. \end{aligned}$$

To calculate the image of \mathcal{D} under ϱ , we use the Wu formulas. For $u \geq 2$

$$\begin{aligned} \varrho \circ \text{Sq } w_u &= \varrho \left(\sum_{d=0}^u \sum_{j=0}^d \binom{u-d+j-1}{j} w_{u+j} w_{d-j} \right) = \\ &= \sum_{d=0}^u \left(\binom{u-1}{d} x t^{u+d-2} + \sum_{j=0}^{d-2} \binom{u-d+j-1}{j} x^2 t^{u+d-4} \right) = \\ &= \sum_{d=0}^u \left(\binom{u-1}{d} x t^{u+d-2} + \binom{u-2}{d-2} x^2 t^{u+d-4} \right) = \\ &= x t^{u-2} (t+1)^{u-1} + x^2 t^{u-2} (t+1)^{u-2} = x t^{u-2} (t+1)^{u-2} (x+t+1). \end{aligned}$$

Similarly,

$$\varrho(w^{-1}) = (1 + x(1+t+t^2+\dots))^{-1} = \frac{t+1}{x+t+1},$$

hence for $s, m \geq 0$, $s + 2m + 2 \leq n - 1$ and a monomial $p \in \mathbb{Z}_2[w_2, \dots, w_{n-1}]$ of degree $2m + 2 + s$ and length $|p| = m + 1$ the corresponding element $[w^{-1} \cdot \text{Sq } p]_{\deg=n}$ of \mathcal{D} is mapped by ϱ to

$$\begin{aligned} R(s, m) &:= \varrho([w^{-1} \text{Sq } p]_{\deg=n}) = \left[\frac{t+1}{x+t+1} x^{m+1} t^s (t+1)^s (x+t+1)^{m+1} \right]_{\deg=n} = \\ &= [(t+1)^{s+1} t^s (x+t+1)^m x^{m+1}]_{\deg=n}. \end{aligned}$$

Note that if a monomial p is divisible by w_1 , then $\text{Sq } p$ and $[w^{-1} \text{Sq } p]_{\deg=n}$ are also divisible by w_1 , consequently $\varrho([w^{-1} \text{Sq } p]_{\deg=n}) = 0$.

Separating the expression for $R(s, m)$ by degree of x we get

$$\begin{aligned} R(s, m) &= \left[\sum_{i=0}^m \binom{m}{i} x^{m+1+i} t^s (t+1)^{s+1+m-i} \right]_{\deg=n} = \\ &= \sum_{i=0}^m \binom{m}{i} \binom{s+1+m-i}{n-2m-2i-2-s} x^{m+1+i} t^{n-2m-2i-2}. \end{aligned}$$

Recall that we use the convention that $\binom{\alpha}{\beta} = 0$ if $\beta < 0$ or $\alpha < \beta$. Note that the binomial coefficient in the above sum is equal to 0 if the exponent of t is negative. When p is the constant 1, we have

$$\begin{aligned} R_0 &:= \varrho([w^{-1} \text{Sq } 1]_{\deg=n}) = \left[\frac{t+1}{x+t+1} \right]_{\deg=n} = \left[\frac{1}{1 + \frac{x}{1+t}} \right]_{\deg=n} = \\ &= \sum_{1 \leq j \leq n/2} x^j [(1+t)^{-j}]_{\deg=n-2j} = \sum_{1 \leq j \leq n/2} \binom{n-j-1}{n-2j} x^j t^{n-2j}. \end{aligned}$$

Let V_R denote the set $\{(s, m) : s, m \geq 0, s+2m+2 \leq n-1\}$. Therefore $\varrho(\mathcal{D})$ is equal to the linear span of the set $\{R(s, m) : (s, m) \in V_R\} \cup \{R_0\}$ in $\mathbb{Z}_2[x, t]$. Denote the linear span of $\{R(s, m) : (s, m) \in V_R\}$ by \mathcal{R}_+ and denote $\varrho(\mathcal{D})$ by \mathcal{R} .

5.1.2. The dimension of \mathcal{R} . The space \mathcal{R} is contained in $\text{im } \varrho_n$, that is, \mathcal{R} is contained in $\langle xt^{n-2}, \dots, x^{\lfloor n/2 \rfloor} t^{n-2\lfloor n/2 \rfloor} \rangle$. We will check whether the monomials $xt^{n-2}, \dots, x^{\lfloor n/2 \rfloor} t^{n-2\lfloor n/2 \rfloor}$ are contained in \mathcal{R} separately in the cases of odd and even n . We will use the criterion of [Gl99] cited above, which states that $\binom{b}{a}$, $0 \leq a \leq b$, is even if and only if there is a binary position at which a has the digit 1 and b has the digit 0.

Lemma 5.12. *The binomial coefficients $\binom{K-p}{p}$ with $0 < p \leq \frac{K}{2}$ are all even if and only if $K+1$ is a power of 2.*

Proof. If $K+1$ is a power of 2, then K written in binary contains only digits 1, hence p and $K-p$ are complementary to each other. Since $p \neq 0$, there is a digit 1 in its binary representation, thus in the same position $K-p$ has digit 0 and the criterion of [Gl99] implies that $\binom{K-p}{p}$ is even. Conversely, if K contains the bit pattern ...10... at position h , say, then $\binom{K-2^h}{2^h}$ is odd by the same criterion. \square

Due to our convention, this result implies that $K+1$ is a power of 2 if and only if the binomial coefficients $\binom{K-p}{p}$ are even for all $p > 0$.

5.1.3. Case of n even. For $n=2$, we have $R_0 = x$, hence $\text{im } \varrho_n = \langle x \rangle = \mathcal{R}$.

For $n > 2$, note that the monomial xt^{n-2} occurs as a summand in $R(s, m)$ only in the case $m=0$ and $R(s, 0) = \binom{s+1}{n-2-s} xt^{n-2}$. If $n \geq 3$, then for $s = n-3$ we have $0 < n-2-s \leq (n-1)/2$ and $(s, 0) \in V_R$. If n is not a power of 2, then we apply Lemma 5.12 with $K = n-1$ and $p = n-2-s$, and obtain that the coefficient of xt^{n-2} in $R(s, 0)$ is not 0. If n is a power of 2, then Lemma 5.12 with the same choice of $K = n-1$ and $p = n-2-s$ implies that $R(s, 0) = 0$ for all $(s, 0) \in V_R$. Hence $xt^{n-2} \in \mathcal{R}_+$ if and only if n is not a power of 2. Note that if $xt^{n-2} \notin \mathcal{R}_+$, then xt^{n-2} does not appear as a summand in any elements of \mathcal{R}_+ .

If $n=4$, then $V_R = \{(0, 0), (1, 0)\}$. We have $R(0, 0) = 0$ as one can check easily and above we showed that $R(1, 0) = 0$, thus \mathcal{R}_+ consists only of the zero element.

Next consider $m = 1, 3, \dots, \frac{n-4}{2}$ for $n \geq 4$ if $4 \nmid n$ and $m = 1, 3, \dots, \frac{n-6}{2}$ for $n \geq 6$ if $4 \mid n$. Choosing $s = n - 2m - 3$ gives us $(s, m) \in V_R$ and $R(n - 2m - 3, m) = \binom{m}{0} \binom{n-m-2}{1} x^{m+1} t^{n-2m-2}$. Since $n - m - 2$ is odd, $R(n - 2m - 3, m) = x^{m+1} t^{n-2m-2}$. Therefore $x^{m+1} t^{n-2m-2}$ is in \mathcal{R}_+ . Setting $s = n - 2m - 4$ gives $(s, m) \in V_R$ and

$$R(n - 2m - 4, m) = \binom{m}{0} \binom{n-m-3}{2} x^{m+1} t^{n-2m-2} + \binom{m}{1} \binom{n-m-4}{0} x^{m+2} t^{n-2m-4}.$$

The first summand is in \mathcal{R}_+ by the argument above, thus so is the second one, which equals to $x^{m+2} t^{n-2m-4}$ due to m being odd. Therefore we obtain that for $n \geq 4$, $4 \nmid n$ the monomials $x^2 t^{n-4}, \dots, x^{n/2}$ are in \mathcal{R}_+ , and for $n \geq 6$, $4 \mid n$ the monomials $x^2 t^{n-4}, \dots, x^{(n-2)/2} t^2$ are in \mathcal{R}_+ .

The only monomial not covered by the cases detailed above is $x^{n/2}$ in the case when n is divisible by 4 and $n \geq 6$. Since all the other monomials either belong to \mathcal{R}_+ or do not appear as summands in any $R(s, m)$ for $(s, m) \in V_R$, we have that $x^{n/2} \in \mathcal{R}_+$ if and only if $x^{n/2}$ occurs as a summand in an $R(s, m)$, $(s, m) \in V_R$. The coefficient of $x^{n/2}$ in $R(s, m)$ can be nonzero only when $n = 2m + 2i + 2$ for some $0 \leq i \leq m$ and $s = 0$, and then the coefficient is

$$\binom{m}{i} \binom{s+1+m-i}{n-2m-2i-2-s} = \binom{m}{\frac{n}{2}-m-1} \binom{2m+2-\frac{n}{2}}{0} = \binom{m}{\frac{n}{2}-m-1},$$

and we have $(s, m) \in V_R$ if and only if $m \geq i \geq 1$, $n \geq 6$. By Lemma 5.12, $\binom{m}{\frac{n}{2}-m-1}$ is even for all possible m exactly when $\frac{n}{2}$ is a power of 2.

To summarize, when n is even, the set $\{R(s, m) : (s, m) \in V_R\}$ generates the space $\text{im } \varrho_n = \langle x t^{n-2}, \dots, x^{n/2} \rangle$ if n is not a power of 2. If n is a power of 2, then we know that \mathcal{R}_+ is spanned by all monomials in $\text{im } \varrho$ of degree n except for $x t^{n-2}$ and $x^{n/2}$. Let us check the coefficients of $x t^{n-2}$ and $x^{n/2}$ in $R_0 = \binom{n-2}{n-2} x t^{n-2} + \dots + \binom{n-\frac{n}{2}-1}{0} x^{n/2}$. Both of their coefficients are 1, hence $\mathcal{R} = \langle \mathcal{R}_+ \cup \{R_0\} \rangle$ has codimension 1 in $\text{im } \varrho_n$ and $\text{im } \varrho_n / \mathcal{R}$ is spanned by $x t^{n-2} + \mathcal{R} = x^{n/2} + \mathcal{R}$.

5.1.4. *Case of n odd.* Let us call a monomial $x^h t^{n-2h}$ *admissible* if h is not a power of 2.

Lemma 5.13. *For an admissible monomial $x^h t^{n-2h}$ with $1 \leq h \leq \frac{n-1}{2}$ and $2^u < h < 2^{u+1}$, where $u \geq 0$, there exist an integer $r(h)$, a set of integers E_h with $2^u \leq \alpha \leq h - 1$ for all $\alpha \in E_h$, and an element $R_h \in \mathcal{R}_+$ such that $R_h + x^h t^{n-2h}$ is a linear combination of monomials $x^\alpha t^{n-2\alpha}$, $\alpha \in E_h$.*

Proof. Take the greatest $r = r(h) \geq 0$ for which $2^r \mid h$. Note that

$$(5.1) \quad h - 1, \dots, h - 2^r \geq 2^u$$

since $h - 2^r$ has the same binary form as h except for the least significant digit 1, which is changed to 0. Also note that $h \equiv 2^r \pmod{2^{r+1}}$ and $h \geq 2^{r+1} + 2^r$. Consider $R(n - 2h, h - 1 - 2^r)$. This polynomial is in \mathcal{R}_+ since $(n - 2h, h - 1 - 2^r) \in V_R$. Indeed, $h - 1 \geq 2^{r+1} > 2^r$, $2h < n$ and $n - 2h + 2(h - 1 - 2^r) + 2 = n - 2^{r+1} < n$. We have

$$\begin{aligned} R(n - 2h, h - 1 - 2^r) &= \binom{h-1-2^r}{0} \binom{n-h-2^r}{2^{r+1}} x^{h-2^r} t^{n-2h+2^{r+1}} + \dots \\ &\quad \dots + \binom{h-1-2^r}{2^r} \binom{n-h-2^{r+1}}{0} x^h t^{n-2h}. \end{aligned}$$

The coefficient of the last monomial of this sum is nonzero because $h \geq 2^{r+1} + 2^r$, $n - h - 2^{r+1} > n - 2h > 0$, and $h - 2^r - 1 \equiv -1 \pmod{2^{r+1}}$ implies that the r -th binary digit of $h - 2^r - 1$ is 1.

Let R_h be $R(n - 2h, h - 1 - 2^r)$ and let E_h be $\{h - 1, \dots, h - 2^r\}$. Then, by (5.1) we have the statement. \square

Proposition 5.14. *Let $1 \leq h \leq \frac{n-1}{2}$ and assume that $u \geq 0$ is the greatest integer such that $2^u \leq h$. Then $x^h t^{n-2h} \in \mathcal{R}_+$ or $x^h t^{n-2h} + x^{2^u} t^{n-2^{u+1}} \in \mathcal{R}_+$ holds.*

Proof. If h is a power of 2, then the statement obviously holds. Hence we can assume that $x^h t^{n-2h}$ is admissible and $2^u < h < 2^{u+1}$. Apply Lemma 5.13 to $x^h t^{n-2h}$, then $x^h t^{n-2h} + R_h$ is a linear combination of monomials, where the exponents $\alpha \in E_h$ satisfy $2^u \leq \alpha \leq h - 1$. Let $E'_h = \{\alpha \in E_h : \alpha > 2^u\}$.

Again, if $h - 1 > 2^u$ and $E'_h \neq \emptyset$, then apply Lemma 5.13 to the admissible monomials of the linear combination $x^h t^{n-2h} + R_h$, then we obtain that $x^h t^{n-2h} + R_h + \sum_{\alpha \in E'_h} R_\alpha$ is a linear combination of monomials whose degree in x is at least 2^u and smaller than $h - 1$. Again, if $h - 2 > 2^u$ and there are resulting admissible monomials in the last linear combination, then apply Lemma 5.13, and iterate this procedure until we get that $x^h t^{n-2h} + \tilde{R} = \varepsilon x^{2^u} t^{n-2^{u+1}}$, where $\tilde{R} \in \mathcal{R}_+$ and $\varepsilon \in \{0, 1\}$. Note that the procedure finishes in a finite number of steps since at each step the linear combination of the next step has smaller degrees in x , while a common lower limit for the degrees of x is 2^u . This proves our claim. \square

Proposition 5.15. *Let $1 \leq h \leq \frac{n-1}{2}$. For every $r \geq 0$, if $x^h t^{n-2h} \notin \mathcal{R}_+$ and $n - 2h \geq 2^r - 1$, then $2^r | n - h + 1$.*

Proof. The proof will proceed by induction on r , with $r = 0$ as the trivial starting case: $1 | n - h + 1$ always holds.

Let $r \geq 1$ and suppose that the statement holds for $r - 1$. Let h be such that $x^h t^{n-2h} \notin \mathcal{R}_+$ and $n - 2h \geq 2^r - 1$. Assume indirectly that $2^r \nmid n - h + 1$. We have $n - 2h \geq 2^r - 1 > 2^{r-1} - 1$ hence by the induction hypothesis we have $n - h + 1 \equiv 2^{r-1} \pmod{2^r}$.

Consider

$$(5.2) \quad R(n - 2h - 2^{r-1}, h - 1) = \binom{h-1}{0} \binom{n-h-2^{r-1}}{2^{r-1}} x^h t^{n-2h} + \dots \\ \dots + \binom{h-1}{\lfloor 2^{r-2} \rfloor} \binom{n-2h-2^{r-2}}{2^{r-1}-2\lfloor 2^{r-2} \rfloor} x^{h+\lfloor 2^{r-2} \rfloor} t^{n-2h-2\lfloor 2^{r-2} \rfloor},$$

where taking the integral part of 2^{r-2} is only needed to handle the case of $r = 1$. Since $h \geq 1$, $n - 2h - 2^{r-1} \geq 2^r - 1 - 2^{r-1} = 2^{r-1} - 1 \geq 0$ and $n - 2h - 2^{r-1} + 2(h - 1) + 2 = n - 2^{r-1} < n$, we have that $R(n - 2h - 2^{r-1}, h - 1) \in \mathcal{R}_+$.

For $y = h + 1, \dots, h + \lfloor 2^{r-2} \rfloor$ we have $n - 2y \geq n - 2(h + \lfloor 2^{r-2} \rfloor) \geq 2^r - 1 - 2^{r-1} = 2^{r-1} - 1$, and since by the induction hypothesis $2^{r-1} | n - h + 1$, none of the values $n - y + 1$ can be divisible by 2^{r-1} . Applying the induction hypothesis again gives us that all of the monomials in (5.2) except possibly the first one are in \mathcal{R}_+ . But the coefficient of the first term is nonzero: $\binom{h-1}{0} = 1$ and $\binom{n-h-2^{r-1}}{2^{r-1}} = 1$ by [Gl99] since $n - h - 2^{r-1} \equiv 2^r - 1 \pmod{2^r}$ has the binary digit 1 at the only location where 2^{r-1} has a 1. Therefore $R(n - 2h - 2^{r-1}, h - 1) + x^h t^{n-2h} \in \mathcal{R}_+$ and consequently $x^h y^{n-2h} \in \mathcal{R}_+$, finishing the proof. \square

We apply Proposition 5.15 to monomials of the form $x^{2^u} t^{n-2^{u+1}}$, $u \geq 0$, n is odd and $2^{u+1} \leq n$, with the choice of $r \geq 0$ so that $2^{r+1} < n < 2^{r+2}$. Note that $u \leq r$ due to

$2^u \leq \frac{n-1}{2} < 2^{r+1}$. We get that if $x^{2^u} t^{n-2^{u+1}} \notin \mathcal{R}_+$, then at least one of the following has to hold:

- (a) $2^r \mid n - 2^u + 1$. We know that $n - 2^u$ is at least $n - 2^r > 2^{r+1} - 2^r = 2^r$, and on the other hand $n - 2^u$ is at most $n - 1 < 2^{r+2} - 1$. There are only two integers i in the open interval $(2^r, 2^{r+2} - 1)$ which satisfy the divisibility condition $2^r \mid i + 1$, namely $2^{r+1} - 1$ and $3 \cdot 2^r - 1$. Hence $n - 2^u$ is either $2^{r+1} - 1$ or $3 \cdot 2^r - 1$.
- (b) $n - 2^{u+1} < 2^r - 1$. Then $2^{u+1} > n - 2^r + 1 > 2^r$ since $2^{r+1} < n$, and $u \leq r$ implies that $u = r$.

We claim that in the case $n - 2^u = 3 \cdot 2^r - 1$ of (a) the monomial $x^{2^u} t^{n-2^{u+1}}$ is actually in \mathcal{R}_+ . Indeed, check the statement of Proposition 5.15 for $r + 1$. Then $2^{r+1} \nmid n - 2^u + 1 = 3 \cdot 2^r$, and $n - 2^{u+1} = 3 \cdot 2^r - 1 - 2^u \geq 2^{r+1} - 1$ since $2^u \leq 2^r$, therefore we have $x^{2^u} t^{n-2^{u+1}} \in \mathcal{R}_+$.

Hence if $x^{2^u} t^{n-2^{u+1}} \notin \mathcal{R}$, then we are left with two possibilities:

- (a) $n = 2^{r+1} + 2^u - 1$,
- (b) $n - 2^{u+1} < 2^u - 1$ and $2^{u+1} < n < 2^{u+2}$.

In the case (b), note that if $x^{2^u} t^{n-2^{u+1}} \notin \mathcal{R}_+$, then Proposition 5.15 implies that for any $0 \leq r' \leq u$ either

- (i) $n - 2^{u+1} < 2^{r'} - 1$ or
- (ii) $n - 2^u \equiv 2^{r'} - 1 \pmod{2^{r'}}$. Due to $r' \leq u$ this condition is equivalent to $n - 2^{u+1} \equiv 2^{r'} - 1 \pmod{2^{r'}}$.

In the case (b) choose r' to satisfy $2^{r'} \leq n - 2^{u+1} < 2^{r'+1}$. This value of r' will be smaller than u due to $n - 2^{u+1} < 2^u - 1$. For this choice of r' , condition (i) fails, thus condition (ii) has to hold. This implies that $n - 2^{u+1} - (2^{r'} - 1) = l2^{r'}$. This integer l can be only 1 because $n - 2^{u+1} < 2^{r'+1}$. Thus, $n - 2^{u+1} = 2^{r'+1} - 1$.

Therefore if $x^{2^u} t^{n-2^{u+1}} \notin \mathcal{R}_+$, then we have two possible cases:

- (a) $n = 2^{r+1} + 2^u - 1$, where we chose $r \geq 0$ so that $2^{r+1} < n < 2^{r+2}$, this implied $u \leq r$,
- (b) $n = 2^{u+1} + 2^{r'+1} - 1$, where we chose $r' \geq 0$ so that $2^{r'} \leq n - 2^{u+1} < 2^{r'+1}$, this implied $r' < u$.

By Proposition 5.14 in both cases (a) and (b) we have $\mathcal{R}_+ = \text{im } \varrho_n$, unless there are positive integers $a > b$ such that $n = 2^a + 2^b - 1$. Moreover in these exceptional cases, when $n = 2^a + 2^b - 1$ with $a > b > 0$, the linear space \mathcal{R}_+ has to contain all the monomials $x^{2^u} t^{n-2^{u+1}}$ except possibly those with $u = a - 1$ (in case (b)) and $u = b$ (in case (a)).

Proposition 5.16. *If $n = 2^a + 2^b - 1$ with $a > b > 0$ and $u = a - 1$ or $u = b$, then the monomial $x^{2^u} t^{n-2^{u+1}}$ does not appear as a summand in any $R(s, m) \in \mathcal{R}_+$.*

Proof. In the relation $R(s, m)$ the monomial $x^{2^u} t^{n-2^{u+1}}$ has the coefficient

$$(5.3) \quad \binom{m}{2^u - m - 1} \binom{m + s + 1 - (2^u - m - 1)}{n - 2m - 2(2^u - m - 1) - 2 - s} = \binom{m}{2^u - 1 - m} \binom{2m + 2 - 2^u + s}{n - 2^{u+1} - s}.$$

The first binomial coefficient is even unless $2^u = m + 1$ according to Lemma 5.12 for $K = 2^u - 1$, hence we only consider the case $m = 2^u - 1$. Then the second binomial coefficient becomes $\binom{2^u + s}{n - 2^{u+1} - s}$ with s running from 0 to $n - 2m - 3 = n - 2^{u+1} - 1$ as follows from the condition $(s, m) \in V_R$. For $u = b$ we have $n - 2^u = 2^a - 1$ and Lemma 5.12 with $K = n - 2^u$ implies that all of these coefficients (5.3) are even. For $u = a - 1$ we have $2^{u+1} + s \geq 2^{u+1} = 2^a$ and

$n - 2^{u+1} - s \leq n - 2^a = 2^b - 1 < 2^{a-1}$, thus the criterion of [Gl99] gives the same results for $\binom{2^u+s}{n-2^{u+1}-s}$ and $\binom{2^u+s-2^{a-1}}{n-2^{u+1}-s} = \binom{s}{2^b-1-s}$. Since $n - 2^{u+1} - s \geq 1$, Lemma 5.12 proves that $\binom{s}{2^b-1-s}$ is even for all choices of s , as claimed. \square

This means that if $n = 2^a + 2^b - 1$, $a > b > 0$, then the monomials $x^{2^{a-1}}t^{n-2^a}$ and $x^{2^b}t^{n-2^{b+1}}$ never appear as summands in any $R(s, m) \in \mathcal{R}_+$ and hence the algorithm of Proposition 5.14 leads to $x^h t^{n-2^h} \in \mathcal{R}_+$ when $u = a - 1$ or $u = b$. Consequently, \mathcal{R}_+ is spanned by all monomials from xt^{n-2} to $x^{\frac{n-1}{2}}t$ except $x^{2^{a-1}}t^{n-2^a}$ and $x^{2^b}t^{n-2^{b+1}}$, which span a linear space complementary to \mathcal{R}_+ .

To summarize, when n is odd, then we have three possibilities:

- If $n \neq 2^a + 2^b - 1$ for any $a > b > 0$, then $\mathcal{R}_+ = \text{im } \varrho_n$.
- If $n = 3 \cdot 2^b - 1$ for some $b > 0$, then

$$\mathcal{R}_+ = \langle xt^{n-2}, \dots, x^{2^b-1}t^{n-2^{b+1}+2}, x^{2^b+1}t^{n-2^{b+1}-2}, \dots, x^{\frac{n-1}{2}}t \rangle.$$

- If $n = 2^a + 2^b - 1$ for some $a > b + 1$, $b > 0$, then

$$\begin{aligned} \mathcal{R}_+ = \langle xt^{n-2}, \dots, x^{2^b-1}t^{n-2^{b+1}+2}, x^{2^b+1}t^{n-2^{b+1}-2}, \dots, \\ x^{2^{a-1}-1}t^{n-2^a+2}, x^{2^{a-1}+1}t^{n-2^a-2}, \dots, x^{\frac{n-1}{2}}t \rangle. \end{aligned}$$

Finally, the relation R_0 contains the monomials $x^{2^{a-1}}t^{n-2^a}$ and $x^{2^b}t^{n-2^{b+1}}$ with the coefficients $\binom{n-2^{a-1}-1}{n-2^a} = \binom{2^b+2^{a-1}-2}{2^b-1}$ and $\binom{n-2^b-1}{n-2^{b+1}} = \binom{2^a-2}{2^a-2^{b+1}-1}$, both of which are of the form $\binom{\text{even}}{\text{odd}}$ and thus are even by the criterion of [Gl99]. Hence $\mathcal{R}_+ = \mathcal{R}$.

5.1.5. Proofs of the statements.

Proof of Theorem 3.12. By the above, if $n \neq 2^a + 2^b - 1$, $a > b \geq 0$, then any oriented n -manifold which has a fold map in codimension -1 is unoriented null-cobordant. Unless n is divisible by 4, this implies that M is also oriented null-cobordant, see [Wa60].

In the case of $n = 2^a + 2^b - 1$, $a > b \geq 0$, the Stiefel-Whitney characteristic numbers of M which belong to the complete preimage $\varrho^{-1}(\mathcal{R})$ have to vanish. This leaves the following possibilities for nonzero characteristic numbers:

- if n is a power of 2, then $\varrho^{-1}(\mathcal{R})$ is spanned by $w_n + w_2^{n/2}$ and all monomials except w_n and $w_2^{n/2}$. In this case $[M] \in \mathfrak{A}^1$.
- if $n = 2^a + 2^b - 1$, $a > b > 0$, then $\varrho^{-1}(\mathcal{R})$ is spanned by all monomials of length not equal to either 2^b or 2^{a-1} as well as the relations in \mathcal{I} corresponding to these exceptional lengths. When $a = b + 1$, the two lengths coincide and we get that $[M] \in \mathfrak{B}^1$, while in the other case we get that $[M] \in \mathfrak{C}^2$.

In the remaining case of n divisible by 4, we need to additionally calculate the Pontryagin characteristic numbers of M to determine its oriented cobordism class. Theorem 3.5 shows that all the rational Pontryagin classes of M except $p_1^{\mathbb{Q}}(TM)$ have to vanish, hence the only Pontryagin number that may be nonzero is $p_1^{n/4}[M]$. If we additionally assume that M is unoriented null-cobordant, then this number has to be even as its reduction modulo 2 is the Stiefel-Whitney characteristic number $w_2^{n/2}[M]$. \square

Proof of Theorem 3.16. By Proposition 3.7 we know that $TM \oplus \varepsilon^1 = \zeta^2 \oplus \varepsilon^{n-1}$ for some 2-dimensional bundle ζ . Hence $w(TM) = 1 + w_1(\zeta) + w_2(\zeta)$, and we will denote the characteristic class $w_i(\zeta)$ by w_i for brevity. The only nonzero total Steenrod squares of these classes are $Sq(w_1) = w_1(1 + w_1)$ and $Sq(w_2) = w_2 + w_2w_1 + w_3 + w_2^2 = w_2(1 + w_1 + w_2)$. Thus, we

can compute the Dold relation corresponding to the polynomial $w_1^a w_2^b$ with $a + 2b \leq n - 1$, $a, b \geq 0$, as the degree n part of the expression

$$\frac{w_1^a (1 + w_1)^a w_2^b (1 + w_1 + w_2)^b}{1 + w_1 + w_2} = w_1^a (1 + w_1)^a w_2^b (1 + w_1 + w_2)^{b-1}.$$

Setting $b = 0$, $0 \leq a \leq n - 1$ gives

$$\begin{aligned} R(a) &:= \left[\frac{w_1^a (1 + w_1)^a}{1 + w_1 + w_2} \right]_{\deg=n} = \left[\frac{w_1^a (1 + w_1)^{a-1}}{1 + \frac{w_2}{1+w_1}} \right]_{\deg=n} = \\ &= \left[w_1^a (1 + w_1)^{a-1} \sum_{j=0}^{\infty} \frac{w_2^j}{(1 + w_1)^j} \right]_{\deg=n} = \sum_{j=0}^{\frac{n-a}{2}} w_1^a w_2^j [(1 + w_1)^{a-1-j}]_{\deg=n-a-2j} = \\ &= \sum_{j=0}^{\frac{n-a}{2}} \binom{a-1-j}{n-a-2j} w_1^{n-2j} w_2^j. \end{aligned}$$

Here we use the analytical definition of binomial coefficients: $\binom{u}{v} = 0$ if $v < 0$, $\binom{u}{0} = 1$ and $\binom{u}{v} = \frac{u(u-1)\dots(u-v+1)}{v!}$ in the other cases. Choosing $a = n - 2m$ for any $0 < m \leq n/2$ we get

$$\begin{aligned} R(n - 2m) &= \sum_{j=0}^m \binom{n-2m-1-j}{2m-2j} w_1^{n-2j} w_2^j = \\ &= \binom{n-2m-1}{2m} w_1^n + \dots + \binom{n-3m-1}{0} w_1^{n-2m} w_2^m \end{aligned}$$

with analytical binomial coefficients. Here the exponent of w_2 in all the summands except the last one is less than m , and the last summand has coefficient 1. Therefore for all $0 < m \leq n/2$ the monomial $w_2^m w_1^{n-2m}$ is linearly dependent on the monomials with smaller exponents of w_2 and $R(n - 2m)$. Consequently, all the monomials $w_2 w_1^{n-2}, \dots, w_2^m w_1^{n-2m}$ are linearly dependent on w_1^n and $\{R(n - 2m) : 1 \leq m \leq n/2\}$. Evaluating these classes on the fundamental class of M , we get that all Stiefel-Whitney characteristic numbers of $[M]$ depend linearly on $w_1^n[M]$ (with coefficients depending only on n). This condition either defines the 1-dimensional linear subspace $\mathfrak{D}^1 \leq \mathfrak{N}_n$ or implies that M is unoriented null-cobordant. \square

Proof of Proposition 3.17. Choose any index set $I = (i_1, \dots, i_r)$ such that $r = |I| \geq 2$ and $\sum_{j=1}^r i_j = n$. If for any j we have $i_j \leq k$, then $w_I = 0$ due to the vanishing condition imposed on the Stiefel-Whitney classes of TM . If $i_j \geq k + 1$ for $j = 1, \dots, r$, then for $J = (n - (r - 1)(k + 1), k + 1, \dots, k + 1)$ the characteristic numbers $w_I[M]$ and $w_J[M]$ coincide by Corollary 5.11. But $w_{k+1}(TM) = 0$ by [MS74, Problem 8-B], implying that $w_J[M] = 0$ and thus $w_I[M] = 0$ whenever $|I| \geq 2$. \square

Proof of Proposition 3.18. Perturb the Morin map to get a cusp map [Sad03]. By Proposition 5.10 and [MS74, Problem 8-B] the statement follows similarly to the previous one, details are left to the reader. \square

REFERENCES

- [An85] Y. Ando, On the elimination of Morin singularities. J. Math. Soc. Japan **37** (1985), no. 3, 471–487.
- [An87] ———, On the higher Thom polynomials of Morin singularities, Publ. RIMS, Kyoto Univ. **23** (1987), 195–207.

- [An01] ———, Folding maps and the surgery theory on manifolds, *J. Math. Soc. Japan* **53** (2001), no. 2, 357–382.
- [An04] ———, Existence theorems of fold maps, *Japan J. Math.* **30** (2004), 29–73.
- [An07] ———, A homotopy principle for maps with prescribed Thom-Boardman singularities, *Trans. Amer. Math. Soc.* **359** (2007), 489–515.
- [At61] M. F. Atiyah, Immersions and embeddings of manifolds, *Topology* **1** (1961), 125–132.
- [BH04] A. Banyaga and D. Hurtubise, A proof of the Morse-Bott lemma, *Expo. Math.* **22** (2004), 365–373.
- [Boa67] J. M. Boardman, Singularities of differentiable maps, *IHES Publ. Math.* **33** (1967), 21–57.
- [Bott70] R. Bott, On a topological obstruction to integrability, *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968)* pp. 127–131 Amer. Math. Soc., Providence, R.I., 1970.
- [Ch83] D. S. Chess, A note on the classes $[S_1^k(f)]$, *Proc. Symp. Pure Math.* **40** (1983), 221–224.
- [Do56] A. Dold, Erzeugende der Thomschen Algebra \mathfrak{N} , *Math. Z.* **65** (1956), 25–35.
- [Ga78] A. M. Gabrielov, Combinatorial formulas for Pontryagin classes and GL-invariant chains. (Russian) *Funktsional. Anal. i Prilozhen.* **12** (1978), no. 2, 1–7, 95.
- [Gl99] J. W. L. Glaisher, On the residue of a binomial-theorem coefficient with respect to a prime modulus, *Quart. J. Pure App. Math.* **30** (1899), 150–156.
- [GG73] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Math. 14, Springer-Verlag, New York, 1973.
- [GP07] H. Geiges and F. Pasquotto, A formula for the Chern classes of symplectic blow-ups, *J. London Math. Soc.* **76** (2007), 313–330.
- [MS74] J. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. of Math. Studies, No. 76, Princeton Univ. Press, Princeton, N. J.; Univ. of Tokyo Press, Tokyo, 1974.
- [OSS03] T. Ohmoto, O. Saeki and K. Sakuma, Self-intersection class for singularities and its application to fold maps, *Trans. Amer. Math. Soc.* **355** (2003), 3825–3838.
- [Po47] L. S. Pontryagin, Characteristic cycles on differentiable manifolds. (Russian) *Mat. Sbornik N. S.* **21 (63)** (1947), 233–284.
- [Ro52] V. A. Rohlin, Intrinsic definition of Pontryagin’s characteristic cycles. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **84** (1952), 449–452.
- [Sad03] R. Sadykov, The Chess conjecture, *Alg. Geom. Topol.* **3** (2003), 777–789.
- [SSS10] R. Sadykov, O. Saeki and K. Sakuma, Obstructions to the existence of fold maps, arXiv:1003.2754.
- [Sae92] O. Saeki, Notes on the topology of folds, *J. Math. Soc. Japan* **44** (1992), 551–566.
- [SS98] O. Saeki and K. Sakuma, Maps with only Morin singularities and the Hopf invariant one problem, *Math. Proc. Camb. Phil. Soc.* **124** (1998), 501–511.
- [Wa60] C. T. C. Wall, Determination of the cobordism ring, *Ann. Math.* **72** (1960), 292–311.

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